Coalgebraic Structure and Intermediate Hopf-Galois Extensions of Thom Spectra in Quasicategories

by
Jonathan Beardsley

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Abstract

We extend Lurie's work on derived algebraic geometry to define highly structured $\mathbb{E}_n$-coalgebras, bialgebras and comodules in the homotopy theorist's category of spectra. We then show that representable comonads give examples of coalgebras in categories of module spectra for $\mathbb{E}_n$-rings. This immediately leads to an identification of spectral (and more generally quasicategorical) descent data with certain quasicategories of comodules.

Using this new framework we then extend the work of Rognes and Hess to define Hopf-Galois extensions of $\mathbb{E}_n$-ring spectra. We use this machinery to produce many new examples of intermediate Hopf-Galois extensions. Such structures, unlike the intermediate extensions of Galois covers, are not generally controlled by a Galois correspondence. We do however show that intermediate Hopf-Galois extensions are ubiquitous among Thom spectra. Of particular interest are a number of classical cobordism spectra, e.g. $M\mathrm{U}$ and $M\mathrm{Spin}$, that can now be described as quotients of other cobordism spectra, e.g. $M\mathrm{U}$ is the quotient of an action of S^1 on $M\mathrm{SU}$, and $M\mathrm{Spin}$ is the quotient of action of $K(\mathbb{Z}, 3)$ on $M\mathrm{String}$.

Producing such intermediate extensions is accomplished by recognizing the Thom spectrum of a morphism of Kan complexes $f : X \to BGL_1(R)$ as a quotient of R by an action $\Omega X \to GL_1(R)$. As a result, given a fibration $F \to E \to B$ of n-fold loop spaces, and a morphism of n-fold loop spaces $f : E \to BGL_1(R)$, we can produce a sequence of Hopf-Galois extensions $R \to R/\Omega F \to R/\Omega E$. Importantly, the bialgebra associated to the former extension is F and associated to the latter is $E/F = B$, which is distinctly reminiscent of the classical Galois correspondence.

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Contents

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Introduction

Classically, algebraic geometry is the study of commutative rings and their categories of modules (or, similarly, the study of schemes and their categories of sheaves). Algebraic topology is about using algebraic invariants, e.g. rings and modules, to better classify topological spaces. Homotopy theorists work with a category of objects called spectra which, when applied to topological spaces, produce abelian groups of invariants. Since spectra give algebraic objects when applied to spaces, it should not be surprising that algebraic geometry is useful in algebraic topology. If we can understand the algebraic invariants of spaces a little better, then we can understand the spaces a little better, and algebraic geometry provides a powerful kit of tools for understanding the algebraic invariants. Category theory on the other hand provides tools for generalizing familiar structures (e.g. rings and modules) to less familiar settings. In particular, the last twenty years have witnessed the development of a number of (quasi-)categories of spectra which are suitably symmetric monoidal [EKMM95] [HSS00][Lur14]. These categories provide ways of talking about rings and modules in spectra, making it possible to do algebraic geometry directly in the category of spectra. This differs from earlier uses of algebraic geometry in stable homotopy theory in that it attempts to study the "geometry" of the spectra themselves, rather than that of the algebraic invariants they produce. Of course generalization is always a lossy operation, so we can't expect to do everything with spectra that we can do with, say, commutative rings. For instance, there is still not an agreed upon notion of a prime ideal of a ring spectrum. This thesis, however, provides some new examples of strikingly algebraic and heretofore completely unknown structure on ring objects in spectra. Specifically we indicate the ubiquity of Hopf-Galois extensions and coalgebraic structure among Thom spectra. We postpone an intuitive description of Hopf-Galois extensions, or of why the reader should find them interesting, until Section 4.2.1.

We will strongly rely on the existing (and seminal) works of Hess and Rognes on descent and Galois theory in stable homotopy theory [Hes10][Rog08] and the foundational work of Lurie on spectral algebraic geometry [Lur14]. In the above cited documents, especially the first two, the insights of Ravenel, Morava and others begin to be stated in terms of spectra themselves rather than their homotopy groups. We expand upon that work here, and hope to point to new directions of study within which homotopy theory, geometry, arithmetic and possibly even mathematical physics may start to cohabitate more openly.

The focus of much of this thesis is on *coalgebraic structure in quasicategories.* That is, homotopy coherent notions of coalgebras, comodules and bialgebras. These concepts have all been extremely well studied in the discrete case. The book *Corings and Comodules* of Brzezinski and Wisbauer has been constantly inspiring as a reference for discrete coalgebra [BW03]. As the title suggests, a rigorous and thorough theory of corings and comodules is developed therein, including numerous applications to descent theory and the theory of Hopf-Galois extensions of rings. There are many other useful references regarding the aforementioned topics in the discrete setting including [KO74], [BLR90], [MŞ03], [Mes], [Her04], [Sim], [Str04] and [Mon09].

Theories of Hopf-algebras, Hopf-algebroids, their associated comodules and coalgebras, and the homological algebra thereof have been especially important in chromatic homotopy theory and computations of the stable homotopy groups of spheres [Rav86] [Rav92] [Ada69]. All of this work was, however, strictly about *discrete* objects. In other words, the Hopf-algebras, coalgebras, comodules and other related structures were all given as sets or abelian groups with some additional structure. This was partially due to the fact that fully spectral versions of these structures were not necessary to the work being done but also certainly due to the non-existence of symmetric monoidal categories (or quasicategories) of spectra. Now that such categories exist, the work of Ravenel and others can be described in spectra rather than in their homotopy groups. For instance, the E_2-pages of the classical Adams and Adam-Novikov spectral sequences are bigraded derived cotensor products. This thesis makes clear that the appearance of this algberaic cotensor product is a result of these spectral sequences computing the homotopy groups of an honest cotensor product of spectra.

In the rest of Chapter 1, we describe the structure of this thesis, describing the main theorem and giving a table of examples that result from it. We also provide an impressionistic introduction to the theory of quasicategories, within which all the homotopy theory of this thesis takes place. It is the author's hope that this section will make some of the constructions of this work less jarring to the homotopy theorist more familiar with using categories with model structures. We also provide an index explaining all of the notation used in the remainder of the thesis.

In Chapter 2, we review the constructions of Lurie's books *Higher Algebra* [Lur14] and *Higher Topos Theory* [Lur09] which will be necessary to our work. Both of these books are very long and extremely detailed, so we will not come close to fully describing their contents. However, we hope to make their main ideas clear enough that the reader familiar with category and homotopy theory can understand at least the statements of the main results of this thesis. The reader familiar with those two references will find nothing new in this chapter.

Chapter 3 will be devoted to developing useful theories of co- and bialgebras, as well as their associated quasicategories of modules and comodules. Most everything in Chapter 2 is, in one way or another, implicit in [Lur14] and [Lur09]. Moreover, much of it is already well understood by experts. We will, however, for the sake of both the author and the reader, make the relevant constructions explicit, and prove their basic properties. One should also note that useful theories of coalgebraic structure have been recently introduced by Ayala and Francis in the form of n-disk (co)algebras [AF14], and in the work of Gijs Heuts [Heu15]. For the reader not interested in the technical details of highly structured coalgebras in quasicategories, but familiar with the ideas of coalgebras and comodules and general, this chapter is not essential to the understanding of the main results.

In Chapter 4 we will define Hopf-Galois extensions of ring spectra and then investigate examples thereof. We show that a large class of morphisms of ring spectra give examples of Hopf-Galois extensions, but warn that all of our examples are Thom spectra. This should not be surprising to experts since the coactions and torsor conditions required of Hopf-Galois extensions arise very naturally when working with topological spaces and Thom spectrum functors are nothing more than twisted suspension spectrum functors which preserve this structure. Proving the following theorem, which can be used to produce all of the examples of [Rog08] and [Rot09] as well as many new ones, is the main goal of Chapter 4:

Theorem. *Suppose $i : Y \to X$ and $f : X \to BGL_1(\mathbb{S})$ are morphisms of $\mathbb{E}_n$-monoidal Kan complexes for $n > 1$, with X and Y reduced and simply connected. If the composition $X \xrightarrow{f} BGL_1(\mathbb{S}) \to BGL_1(H\mathbb{Z})$ is nullhomotopic then there is a triangle of Hopf Galois extensions of $\mathbb{E}_{n-1}$-monoidal ring spectra, where the associated bialgebras are written over their respective extensions:*

$$\begin{tikzcd} \mathbb{S} \arrow[rr, "\mathbb{S}[X]"] \arrow[dr, "\mathbb{S}[Y]"'] & & Mf \\ & M(f \circ i) \arrow[ur, "\mathbb{S}[X/Y]"'] & \end{tikzcd}$$

To non-experts this theorem is most likely indecipherable at this point, but by the time of its statement in Chapter 4, all of the relevant structure will be well defined. From it we obtain the following table of Hopf-Galois extensions associated to fibrations of loop spaces:

Fibration	Hopf-Galois Extension	Bialgebra
$B\mathrm{SU} \to B\mathrm{U} \to \mathbb{C}P^\infty$	$M\mathrm{SU} \to M\mathrm{U}$	$\mathbb{S}[\mathbb{C}P^\infty]$
$B\mathrm{String} \to B\mathrm{Spin} \to K(\mathbb{Z},4)$	$M\mathrm{String} \to M\mathrm{Spin}$	$\mathbb{S}[K(\mathbb{Z},4)]$
$B\mathrm{U} \to B\mathrm{SO} \to \mathrm{Spin}$	$M\mathrm{U} \to M\mathrm{SO}$	$\mathbb{S}[\mathrm{Spin}]$
$B\mathrm{Sp} \to B\mathrm{SO} \to B(\mathrm{SO}/\mathrm{Sp})$	$M\mathrm{Sp} \to M\mathrm{SO}$	$\mathbb{S}[B(\mathrm{SO}/\mathrm{Sp})]$
$\Omega\mathrm{SU}(n) \to \Omega\mathrm{SU}(n+1) \to \Omega S^{2n+1}$	$X(n) \to X(n+1)$	$\mathbb{S}[\Omega S^{2n+1}]$
$B\mathrm{String} \to B\mathrm{U}[6,\infty) \to B^3\mathrm{Spin}$	$M\mathrm{String} \to M\mathrm{U}[6,\infty)$	$\mathbb{S}[B^3\mathrm{Spin}]$
$B\mathrm{SO} \to B\mathrm{O} \to \mathbb{Z}/2$	$M\mathrm{SO} \to M\mathrm{O}$	$\mathbb{S}[\mathbb{R}P^\infty]$
$\Omega^2 S^3\langle 3\rangle \to \Omega^2 S^3 \to S^1$	$H\mathbb{Z}_2^\wedge \to H\mathbb{Z}/2$	$\mathbb{S}[S^1]$

1.1 A Prelude to Quasicategories

To mathematicians familiar with doing homotopy theory using ordinary or enriched Quillen model categories, some of the below may seem baffling. For instance, how is it that we do not cofibrantly replace anything before taking the smash product, or fibrantly replace a cosimplicial object before taking its totalization? How do we know that these objects, regardless of what "category" in which they reside, are computed in a sufficiently homotopy-invariant way? The fact that we do not need to do these things is perhaps one of the greatest benefits of working with quasicategories and the framework developed by Lurie in [Lur09] and [Lur14]. In other words, very little can be done in a quasicategory in a way that is *not* homotopy invariant.

To understand how quasicategories are blind to homotopy equivalence, it is useful to understand what a quasicategory actually is, and what "homotopy equivalence" means therein. Much of what we describe in this prelude is reviewed in greater detail in the first chapter of this thesis, but we will briefly describe the important points anyway. A quasicategory is not an object that should intimidate any homotopy theorist. It is nothing more than a simplicial set with a certain special property that makes it "behave" like an ordinary category. Recall that what separates an ordinary category from an arbitrary collection of vertices and directed edges is that given three vertices A, B, C and two edges $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$, there is always a third edge $A \xrightarrow{h} C$ yielding a commutative triangle

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] \\ & C. \end{tikzcd}$$

In other words, we can compose morphisms and obtain a new morphism $h = g \circ f$. If we want a simplicial set to behave like a category, with 0-simplices taking the place of objects, and 1-simplices taking the place of morphisms, we have to relax this requirement that $h = f \circ g$ (since what would this really mean within a simplicial set anyway?). So we say that a simplicial set is a quasicategory if for any two 1-simplices $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$, there is another 1-simplex $A \xrightarrow{h} C$ and the data of a 2-simplex whose boundary is the triangle above. This is precisely codified by Definition 2.1.2.1. In particular, notice that we need such "filling cells" in every dimension, since we now have n-morphisms (in the form of n-cells) for all n. This is of course the reason that quasicategories are a model for so-called $(\infty, 1)$-categories and are so often referred to as ∞-categories.

If our morphisms are just 1-simplices, what does it mean then for a morphism to be a homotopy equivalence? It turns out that just like in model categories, quasicategories have *homotopy categories* that can be passed to functorially. Their description is actually quite simple: there is an adjunction between categories and simplicial sets whose right adjoint is given by taking the *nerve*, which should be familiar to category theorists. The left adjoint from simplicial sets to ordinary categories, as in Definition 2.1.2.12, is what we call the homotopy category of a simplicial set (and in particular a quasicategory). As such, a "homotopy equivalence" in a quasicategory is nothing more than a 1-simplex that becomes an isomorphism in the homotopy category.

It is also important to note that model categories always give us quasicategories. Many young homotopy theorists, this one included, have probably wondered "Since we've always got to replace things fibrantly or cofibrantly, why don't we just work in the subcategory of objects which are both fibrant and cofibrant?" The answer, of course, is that many constructions and functors don't preserve this property, so even if we start out with so-called *bifibrant* objects, we may have to fibrantly or cofibrantly replace our objects again in the future. However, when we're working with quasicategories, all constructions and functors are homotopy invariant. As a result, to go from working in a simplicial model category to working in a quasicategory, we take the subcategory of bifibrant objects, take its homotopy coherent nerve, and then use the machinery of quasicategories to continue doing homotopy theory. This is essentially the answer to the above (seemingly naïve) question. This is also why quasicategorically minded homotopy theorists often refer to the quasicategory (or ∞-category) *underlying* a given model category.

As always, there is a trade-off in gaining complete homotopy invariance–one loses tight control over which objects one is working with. For instance, you may notice in the following document that we refer to *a* colimit of a diagram, rather than *the* colimit. The reason for this is that there may be many equivalent colimits of a given diagram and we do not have control over which one we're working with (although we do have control over its homotopy type). Also notice that to say that a quasicategory is symmetric monoidal requires quite a bit more data than is typically necessary for saying an ordinary category is symmetric monoidal. In other words, since we are constantly dragging around *all* the data of the relevant homotopies, being symmetric monoidal is indistinguishable from being $\mathbb{E}_\infty$. So an algebra is not just an object with an associative and unital map $A \otimes A \to A$, but rather all the maps $A^{\otimes n} \to A$ and all their possible symmetries, as indexed by a simplicial set which is equivalent to the nerve of the category of finite pointed sets.

Indeed, the whole area of multiplicative structure in quasicategories, whose technical details are covered below in Section 2.2.2, may be difficult for the newcomer to quasicategories to digest in one pass. A useful example to keep in mind is Segal's idea of Γ-spaces [Seg74]. With Γ-spaces, one notices that all possible symmetries of a commutative multiplication map $A \otimes A \to A$ are already encoded in the category of finite pointed sets $\mathcal{F}in_*$. This is why one way of defining a symmetric monoidal quasicategory is to equate it to a functor $N(\mathcal{F}in_*) \to q\mathcal{C}at$, where the left hand side is the nerve of finite pointed sets (pointedness being required because symmetric monoidal quasicategories are also pointed). Similarly, there is an ordinary category whose nerve, which we will denote by $\mathcal{A}ss^{\otimes}$, parameterizes associative multiplicative structure (up to coherent homotopy). We will also discuss a sequence of multiplicative structures that interpolate between $\mathcal{A}ss^{\otimes}$ and $N(\mathcal{F}in_*)$ called $\mathbb{E}_n^{\otimes}$ for $0 \le n \le \infty$ (where $\mathbb{E}_1^{\otimes} \simeq \mathcal{A}ss^{\otimes}$ and $\mathbb{E}_\infty \simeq N(\mathcal{F}in_*)$).

1.2 Notation

Ab:	the ordinary category of abelian groups and abelian group homomorphisms.
$BGL_1(R)$:	the quasicategory of one dimensional free R-modules and equivalences.
${}^{m}BiAlg_n(\mathcal{C})$:	the quasicategory of co-$\mathbb{E}_m$-$\mathbb{E}_m$-bialgebras in $\mathcal{C}$.
$\mathcal{C}$:	a generic quasicategory.
$\mathcal{C}^{\otimes}$:	a generic $\mathcal{O}$-monoidal category for some ∞-operad $\mathcal{O}^{\otimes}$.
$\mathcal{C}^{\otimes}_{\langle n \rangle}$:	for a fibration $\mathcal{C}^{\otimes} \to \mathcal{F}in_*$, the fiber over $\langle n \rangle$.
$CoAlg_{\mathbb{E}_k}(\mathcal{C})$:	the quasicategory of $\mathbb{E}_k$-coalgebras in $\mathcal{C}$.
$CRng$:	the ordinary category of commutative rings and ring homomorphisms.

Δ^n: the simplicial set represented by the object $\{0, 1 \ldots, n\}$.

Λ^n_i: the i^{th} horn of Δ^n.

$\mathbb{E}^{\otimes}_n$: the little n-cubes ∞-operad.

$\mathcal{F}in_*$: the simplicial nerve of the category of finite pointed sets.

$F \dashv G$: a pair of adjoint functors of which F is the left adjoint.

$Fun(\mathcal{C}, \mathcal{D})$: the quasicategory of functors between quasicategories

$Fun^L(\mathcal{C}, \mathcal{D})$: the quasicategory of small colimit preserving functors between two quasicategories.

Grp: the ordinary category of groups and group homomorphisms.

$GrRng$: the ordinary category of graded rings and grading preserving ring homomorphisms.

$Hom_C(X, Y)$: the set of morphisms between X and Y in an ordinary category C.

$LFib(K)$: the quasicategory of left fibrations over a simplicial set K.

$LMod_R$: the category or quasicategory of left R-modules, depending on context.

$\langle m \rangle^\circ$: the finite set $\{1, 2, \ldots, m\}$

$\langle m \rangle$: the finite set $\{*, 1, 2, \ldots, m\}$, or $\langle m \rangle \coprod \{*\}$.

$Map_{\mathcal{C}}(X, Y)$: the Kan complex of morphisms between X and Y, objects of a small quasicategory $\mathcal{C}$.

Mf: the Thom spectrum associated to a map $f : X \to BGL_1(R)$.

$Mod^{\mathcal{O}}_A$: the quasicategory of operadic modules over an $\mathcal{O}$-algebra A.

$N(-)$: the ordinary nerve functor

$\mathcal{N}(-)$: the simplicial nerve functor

$\mathcal{O}^{\otimes}$: uppercase calligraphic letters with a superscript $\otimes$ indicate ∞-operads.

Ω^∞: the infinite delooping functor $\mathcal{S} \to \mathcal{T}$.

$q\mathcal{C}at$: the quasicategory of small quasicategories.

R/G: a colimit of a morphism of simplicial sets $BG \to LMod_R$ for R a ring spectrum.

$\mathcal{S}$: the quasicategory of spectra, the stabilization of $\mathcal{T}$.

$\mathbb{S}$: the sphere spectrum.

Σ^∞_+: the suspension spectrum functor $\mathcal{T} \to \mathcal{S}$.

$\mathbb{S}[X]$: for a Kan complex X, the suspension spectrum of X, also $\Sigma^\infty_+ X$.

$sSet$: the ordinary category of simplicial sets and natural transformations.

$\mathcal{T}$: the quasicategory of Kan complexes, sometimes referred to as spaces.

Top: the ordinary category of topological spaces.

2

Background

In this chapter we will review the basic structures necessary to our later work. A great deal of research has been done in the intersection of category theory and homotopy theory and several models have been put forward for studying spectra categorically. The oldest model is probably Boardman's *stable homotopy category*, which is an ordinary category which is *tensor triangulated* [Pup73]. Tensor triangulated categories behave much like the category of abelian groups and have been and continue to be extremely useful in homotopy theory (cf. work of Balmer [Bal05] [Bal12]). One of the shortcomings of this model is that homotopic maps are in fact isomorphic in this category. In other words the data of the homotopy identifying two maps has been lost. While there do exist approachable symmetric monoidal categories of topological spaces in which this data is preserved, an analogous category of spectra did not exist for some time. Later, after Quillen introduced the idea of a model category [Qui67], a number of symmetric monoidal model categories of spectra were developed, including the $\mathbb{S}$-modules of [EKMM95] and the symmetric spectra of [HSS00]. The former was used as the basis of Rognes' seminal work on homotopical Galois theory [Rog08] and the latter was used by Hess in her illuminating investigation of homotopical descent theory [Hes10] . We however will make use of another, very different model of spectra (and of spaces, and categories in general). We make use of the theory of quasicategories, originally called *weak Kan complexes* by Boardman and Vogt [BV73] and later expanded upon by Joyal [Joy08]. We will use the encyclopedic work of Lurie as our basis for homotopy theory. The following review comprises introductory material pieced together from [Lur09] and [Lur14].

2.1 Spaces, Spectra and Quasicategories

Before talking about spectra, a good theory of topological spaces is needed. Because it is technically useful, we will work with simplicial sets and, in particular, Kan complexes. An excellent resource for the homotopy theory of simplicial sets can be found in Goerss and Jardine's book [GJ09]. One of the major benefits of working with simplicial sets and Kan complexes instead of, say, compactly generated Hausdorff spaces, is that much homotopy theory can be reduced to combinatorial and set-theoretic considerations. The reason *why* Kan complexes are a valid model of topological spaces relies on Quillen's theory of model categories which we will not review here. Suffice it to say that, up to a suitable notion of homotopy, there is an equivalence between Kan complexes and compactly generated Hausdorff spaces.

2.1.1 Simplicial Sets

Definition 2.1.1.1 (Simplex Category)**.** Let Δ be the ordinary category whose objects are ordered sets $[n] = \{0, 1, \dots, n\}$ and whose morphisms are (non-strictly) order preserving functions.

Definition 2.1.1.2 (Simplicial Sets)**.** Let $sSet$ be the ordinary category of functors $\Delta^{op} \to Set$ with natural transformations as morphisms.

There are certain morphisms in Δ^{op} called face and degeneracy maps that effectively generate all of the other morphisms of Δ^{op} (whence Theorem 2.1.1.4 below).

Definition 2.1.1.3 (Face and Degeneracy Maps)**.** Let $\tilde{d}_i : [m-1] \to [m]$ be the function in Δ which does not have i in its image. In other words, $\tilde{d}_i(k) = k$ for $k < i$ and $\tilde{d}_i(k) = k + 1$ for $k \geq i$. Let the i^{th} face map $d_i : [m] \to [m-1]$ be the morphism in Δ^{op} corresponding to $\tilde{d}_i$.

Let $\tilde{\sigma}_i : [m+1] \to [m]$ be the unique surjection in Δ such that the i^{th} element of $[m]$ has a preimage with exactly 2 elements. In other words $\tilde{\sigma}_i(k) = k$ for $k \leq i$, $\tilde{\sigma}_i(i+1) = i$ and $\tilde{\sigma}_i(k) = k-1$ for $k > i+1$. Let the i^{th} degeneracy map $\sigma_i : [m] \to [m+1]$ be the morphism in Δ^{op} corresponding to $\tilde{\sigma}_i$.

The face and degeneracy map satisfy a certain list of *simplicial identities* given in Figure 1.3 of [GJ09] as well as Proposition 8.1.3 [Wei94]. We will not reproduce them here.

Theorem 2.1.1.4. *The data of a simplicial set $F : \Delta^{op} \to Set$ is entirely determined by a choice of sets $F([m])$ for each m and a choice of face and degeneracy maps $F(d_i)$, $F(\sigma_i)$ for each i and each m that satisfy the simplicial identities.*

Proof. See Chapter 1 of [GJ09] or Proposition 8.1.3 of [Wei94] □

Definition 2.1.1.5 (Function Complexes)**.** Let F and G be simplicial sets. Then define the function complex $Map_{sSet}(F, G)$ by $Map_{sSet}(F, G)([m]) = Hom_{sSet}(F \times \Delta^m, G)$, where the right hand side is the set of natural transformations. Given an order-preserving function $\phi : [m] \to [m']$ in Δ then the function $\phi^* : Map_{sSet}(F \times \Delta^{m'}, G) \to Map_{sSet}(F \times \Delta^m, G)$ is given by

$$(F \times \Delta^{m'} \to G) \mapsto (F \times \Delta^m \overset{1\times\phi}{\to} F \times \Delta^{m'} \to G).$$

Remark 2.1.1.6. It follows immediately that the set of zero simplices of $Map_{sSet}(F, G)$ is in bijection with the set of natural transformations $Hom_{sSet}(F, G)$. In other words, Definition 2.1.1.5 gives $sSet$ the structure of a *closed category* (and as such a simplicially enriched category).

Example 2.1.1.7. Let $\Delta^n : \Delta^{op} \to Set$ be the functor represented by the object $[n]$, i.e. $\Delta^n([m]) = Hom_\Delta([m], [n])$. In terms of the relationship with topological spaces mentioned above, this simplicial set should be thought of as the standard topological n-simplex, i.e. the set of points $(x_0, \ldots, x_n) \in \mathbb{R}^{n+1}$ such that $\sum x_i = 1$ and $x_i > 0$ for all $0 \leq i \leq n$.

Example 2.1.1.8 (Outer and Inner Horns)**.** Let $\Lambda^n_i : \Delta^{op} \to Set$ be the functor that takes $[m]$ to the (order preserving) functions in $Hom_\Delta([m], [n])$ that do not have i in their image. Topologically, this simplicial set should be thought of as the all of the faces of Δ^n except for the i^{th} one. Note that for each $0 \leq i \leq n$ there is a canonical natural transformation $\iota_i : \Lambda^n_i \hookrightarrow \Delta^n$ which is an inclusion on objects. If $i = 0$ or $i = n$ we call Λ^n_i an *outer horn*. Otherwise it is an *inner horn*.

Example 2.1.1.9. The 2-simplex Δ^2 has three horns, one associated to each vertex. Two are outer horns and one is an inner horn. These can be graphically represented as $\angle$, $\wedge$ and $\measuredangle$.

Definition 2.1.1.10 (Kan Complexes)**.** Suppose $F : \Delta^{op} \to Set$ is a simplicial set. Then we say that F is a *Kan complex* if for every solid diagram like the following there is a dotted arrow making the diagram commute:

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook, "\iota_i"] & F \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & \Delta^0 \end{tikzcd}$$

Example 2.1.1.11. Let X be an object in the ordinary category of compactly generated Hausdorff spaces. Then the simplicial set of simplices in X, denoted $Sing(X)$, is a Kan complex.

Remark 2.1.1.12. The way to think about Definition 2.1.1.10 when $n = 2$ is that if one can draw any two legs of a triangle inside of one's would-be space, one can draw the third leg of that triangle and then fill it in a continuous way. Note that the legs of the triangle don't have to be straight, only drawn continuously. In other words, we don't care if the triangle is deformed so long as it is deformed continuously.

2.1.2 Quasicategories

The following definition is due to Joyal [Joy08], and is a slight modification of the original idea of Boardman and Vogt [BV73]

Definition 2.1.2.1 (Quasicategories)**.** Suppose $F : \Delta^{op} \to Set$ is a simplicial set. Then we say that F is a *quasicategory* if for every solid diagram like the following in which $\Lambda^n_i \hookrightarrow \Delta^n$ is the inclusion of an *inner horn* there is a dotted arrow making the diagram commute:

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook, "\iota_i"] & F \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & \Delta^0 \end{tikzcd}$$

Notation 2.1.2.2 (Functors of Quasicategories)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be quasicategories. Then we will denote by $Fun(\mathcal{C}, \mathcal{D})$ the simplicial set (see Definition 2.1.1.5) of natural transformations between $\mathcal{C}$ and $\mathcal{D}$.

Theorem 2.1.2.3. *The simplicial set $Fun(\mathcal{C}, \mathcal{D})$ is a quasicategory.*

Proof. See Proposition 1.2.7.3 of [Lur09]. □

Definition 2.1.2.4 (Mapping Spaces in Quasicategories)**.** Let $\mathcal{C}$ be a quasicategory and let x and y be 0-simplices of $\mathcal{C}$. Note that these induce a 0-simplex inclusion (of constant morphisms) $\{x\} \times \{y\} \hookrightarrow \mathcal{C} \times \mathcal{C}$. Then we define $Map_{\mathcal{C}}(x, y)$ to be the pullback in simplicial sets of the diagram $Map_{sSet}(\Delta^1, \mathcal{C}) \xrightarrow{\pi} \mathcal{C} \times \mathcal{C} \hookleftarrow \{x\} \times \{y\}$.

Theorem 2.1.2.5. *Let F be a quasicategory and x and y be 0-simplices of F. Then $Map_{\mathcal{C}}(x, y)$ is a Kan complex.*

Proof. See Corollary 4.2.1.8 of [Lur09]. In the notation there, our $Map_{\mathcal{C}}(x, y)$ of Definition 2.1.2.4 would be denoted by $Hom_{\mathcal{C}}(x, y)$. □

Remark 2.1.2.6. The fact that the model of mapping complexes internal to a quasicategory given in Definition 2.1.2.4 is equivalent to other models is given in both [Lur14] and [DS11]. Note also that a "morphism" in $Map_{\mathcal{C}}(x, y)$ is only determined up to higher homotopy. We may choose a representative by applying π_0 to the mapping Kan complex.

Example 2.1.2.7. Every Kan complex is a quasicategory. Kan complexes are to quasicategories as groupoids are to categories. In [Lur09] and other references, Kan complexes are often called ∞-groupoids.

Remark 2.1.2.8. Note that the only difference between a Kan complex and quasicategory is that the former has lifts for *all* horn inclusions and the latter has lifts for only the inner horns. The crucial thing to note is that the faces of the simplices of a simplicial set are *ordered*. Thus the definition of a Kan complex says that we can fill in the missing face of a simplex no matter which face is missing and the definition of a quasicategory says we can fill in the missing face *only* if it's not the first or last face. The significance of this in two dimensions is that given the two faces $\{0 \to 1\}$ and $\{1 \to 2\}$ of Δ^2, there is always a face $\{0 \to 2\}$ and moreover, there is a cell filling in the resulting triangle. Compare this to the requirement in an ordinary category that one can always compose morphisms. On the other hand, given the two faces $\{0 \to 1\}$ and $\{0 \to 2\}$, there need not be a face $\{1 \to 2\}$ completing the triangle. This would correspond to the $\{0 \to 2\}$ being invertible, which won't always be the case. However, in a Kan complex, which is supposed to model a topological space, and for which the 1-simplices should correspond to paths in that topological space, it should clearly be true that all paths are invertible.

As a result of work of Joyal, Lurie and many others, one may take the theory of quasicategories as the ambient framework for doing homotopy theory. We take that point of view in this thesis. First, however, we will need to clarify how ordinary categories and model categories fit into the theory of quasicategories.

Definition 2.1.2.9 (Nerve)**.** Let C be a small ordinary category. Define the *nerve* of C, denoted $N(C)$ to be the simplicial set given by $N(C)([m]) = \{$sequences of composable morphisms in C of length $m\}$. The face maps are given by either forgetting the first morphism of such a sequence, composing two of the interior morphisms or forgetting the last morphism. The degeneracy maps are given by insertions of identity morphisms.

Theorem 2.1.2.10. *For an ordinary category C, $N(C)$ is always a quasicategory.*

Proof. We leave it as an exercise to the reader using the definitions given above and [GJ09]. Alternatively see Proposition 1.1.2.2 of [Lur09]. $\square$

Proposition 2.1.2.11. *Taking the nerve of an ordinary category determines a functor* $N(-) : Cat \to sSet$ *which admits a left adjoint* $h : sSet \to Cat$.

Proof. See Proposition 1.2.3.1 of [Lur09]. $\square$

Definition 2.1.2.12 (Homotopy Category)**.** For a simplicial set S, define the *homotopy category* of S to be hS, where h is the left adjoint given in Proposition 2.1.2.11. If S is a quasicategory and f is a 1-simplex of S, we say that f is a homotopy equivalence if it becomes an isomorphism in hS.

Remark 2.1.2.13. If C is a small, simplicially enriched ordinary category then one can produce a quasicategory by taking the nerve $N(C)$ as in Definition 2.1.2.9. However, $N(C)$ only "sees" the 0-simplices of the simplicial mapping sets of C. Thus the quasicategory $N(C)$ is not a good approximation to the homotopy theory of C. The solution to this problem is to use the *simplicial nerve functor* of Cordier [Cor82]. We will *not* review the full construction of this functor here, but refer the reader to Section 1.1.5 of [Lur09].

Notation 2.1.2.14 (Simplicial Nerve)**.** Given a simplicially enriched ordinary category C we denote its simplicial nerve, from Section 1.1.5 of [Lur09], by $\mathcal{N}(C)$.

The following, which is Proposition 1.1.5.10 of [Lur09], gives us that the simplicial nerve is often a quasicategory:

Proposition 2.1.2.15. *Let* C *be a simplicially enriched ordinary category such that for every pair of objects* $X, Y \in C$ *the simplicial set of morphisms* $Map_C(X, Y)$ *is a Kan complex. Then the simplicial nerve* $\mathcal{N}(C)$ *is a quasicategory.*

Proof. See Proposition 1.1.5.10 of [Lur09]. $\square$

Remark 2.1.2.16. The above theorem gives us that any topologically enriched category gives a quasicategory by first applying $Sing(-)$ to mapping spaces and then applying $\mathcal{N}(-)$. If we have a simplicially enriched category whose simplicial sets of morphisms do not form Kan complexes, we may still approximate it with a quasicategory by replacing the simplicial sets of morphisms with weakly equivalent Kan complexes (fibrantly replacing in $sSet$ with the Quillen model structure) then applying $\mathcal{N}(-)$.

Corollary 2.1.2.17. *Let M be a simplicial model category, with full subcategory of fibrant-cofibrant (i.e. bifibrant) objects M°. Then $\mathcal{N}(M^\circ)$ is a quasicategory.*

Proof. This follows from the fact that for a simplicial model category the morphism complex $Map_M(X,Y)$ is a Kan complex whenever X is cofibrant and Y is fibrant. □

Notation 2.1.2.18. For any simplicial model category M we will refer to $\mathcal{N}(M^\circ)$ as the quasicategory *underlying* M. One can check that they do indeed have equivalent homotopy categories.

Remark 2.1.2.19. In the case that M is a model category without a simplicial enrichment, or in other words, in which the simplicial mapping complexes are just given the discrete simplicial set structure, the simplicial nerve will produce a quasicategory whose mapping objects are not of the correct homotopy type. In this case, one must first replace the given model category with a its simplicial localization, as described in [DK80].

Definition 2.1.2.20 (Quasicategory of Quasicategories)**.** Let $q\mathcal{C}at^\Delta$ be the category whose objects are small quasicategories and functors between them. Note that $q\mathcal{C}at^\Delta$ is naturally simplicially enriched. For two quasicategories $\mathcal{C}$ and $\mathcal{D}$, define a new simplicial enrichment of $q\mathcal{C}at^\Delta$ by letting $\widetilde{Map}_{q\mathcal{C}at^\Delta}(\mathcal{C},\mathcal{D})$ be the largest Kan complex contained in the simplicial set of natural transformations $Map_{q\mathcal{C}at^\Delta}(\mathcal{C},\mathcal{D})$ of Definition 2.1.1.5. Define the quasicategory of small quasicategories, denoted $q\mathcal{C}at$, by $\mathcal{N}(q\mathcal{C}at^\Delta)$ where $q\mathcal{C}at^\Delta$ has this new simplicial enrichment.

Remark 2.1.2.21. Note that when we are discussing a simplicial model category M, i.e. a model category M which is compatibly enriched over $sSet$, we often neglect to discuss the model structure on $sSet$, which does indeed play a role in the enrichment. We are always assuming that when $sSet$ is playing the role of enriching another model category, it is equipped with the Quillen model structure (in which the fibrant objects are the Kan complexes). As a result of this assumption, we cannot say that $sSet$ equipped with the Joyal model structure is simplicially enriched. This is the reason that we cannot take $q\mathcal{C}at$ to be $\mathcal{N}(sSet^\circ)$ with $sSet$ having the Joyal model structure. We must instead replace the mapping complexes with their largest sub-Kan-complexes.

Definition 2.1.2.22 (Quasicategory of Spaces)**.** Define $\mathcal{T}$ to be the quasicategory $\mathcal{N}(sSet^\circ)$, where $sSet$ has the Quillen model structure, and as such is enriched over itself.

Definition 2.1.2.23 (Quasicategory of Finite Spaces)**.** Let $sSet^f$ be subcategory of $sSet$ spanned by the simplicial sets with only a finite number of non-degenerate simplices. Define $\mathcal{T}^{fin}$ to be the quasicategory $\mathcal{N}((sSet^f)^\circ)$ where $sSet^f$ has is equipped with the subcategory simplicial model structure.

Remark 2.1.2.24. We will often refer to the 0-simplices of $\mathcal{T}$ and $\mathcal{T}^{fin}$ as spaces and finite spaces, respectively. By making the obvious modifications to the above definitions we also have the quasi-categories of pointed spaces and pointed finite spaces $\mathcal{T}_*$ and $\mathcal{T}_*^{fin}$.

2.1.3 Spectra

We now define the quasicategory of *spectra* and show that it determines the same homotopy theory as classical models. Naïvely a spectrum is a sequence of topological spaces but after Brown's Representability Theorem [Bro62] we know that spectra also represent all cohomology and homology theories in a suitable way. As such we will follow Lurie [Lur14] by defining spectra to be functors $\mathcal{T}_*^{fin} \to \mathcal{T}_*$ satisfying certain properties. Our description of this construction of spectra is incredibly brief, and the interested reader is strongly advised to read the corresponding chapters of [Lur14].

Definition 2.1.3.1. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of quasicategories. Then we say that F is:

1. *Excisive* if $\mathcal{C}$ admits pushouts and F carries pushout squares to pullback squares.

2. *Reduced* if $\mathcal{C}$ admits a final object $*$ and $F(*)$ is a final object of $\mathcal{D}$.

Definition 2.1.3.2 (Spectra)**.** Let $\mathcal{S}$ be the full sub-quasicategory of $Fun(\mathcal{T}_*^{fin}, \mathcal{T}_*)$ spanned by reduced and excisive functors.

Theorem 2.1.3.3. *There is a* suspension spectrum *functor* $\Sigma^\infty : \mathcal{T} \to \mathcal{S}$ *which admits a right adjoint* $\Omega^\infty : \mathcal{S} \to \mathcal{T}$.

Proof. See Proposition 1.4.3.4 of [Lur14]. □

Remark 2.1.3.4. Given a Kan complex X and a spectrum E we can compute the n^{th} E-homology of X by representing X as a colimit of finite spaces (given by attaching cells), applying E to this tower, and computing π_n of the limiting complex. To compute cohomology of a Kan complex X we should compute $\pi_n(Map_{\mathcal{S}}(\Sigma^\infty X, E))$.

Theorem 2.1.3.5. *The homotopy category of $\mathcal{S}$ is a triangulated category and moreover is equivalent to Boardman's stable homotopy category.*

Proof. See Remark 1.4.3.2 and Proposition 1.4.3.6 of [Lur14]. □

Remark 2.1.3.6. The fact that $h\mathcal{S}$ is triangulated comes as a result of it being a stable quasicategory which can be loosely interpreted as saying that the suspension functor is invertible. We will not explore this issue here, but strongly recommend Chapter 1 of [Lur14] for more information.

2.2 Quasicategorical Structures

We wish to be able to talk about algebraic structure in quasicategories, e.g. rings, modules, coalgebras and comodules. Since we are working with quasicategories, we're effectively forced to use the work of Lurie in [Lur14]. However, we should mention that this framework is not the best possible one for every application. For instance, Ravenel and others were able to compute huge swaths of the stable homotopy groups of spheres while using comparatively simple notions of ring spectra and modules over them [Rav86]. Perhaps the crucial thing to understand is that in homotopy theory, when one requires that certain diagrams defining algebraic structure commute, one must decide whether or not those diagrams commute strictly, or only up to homotopy (i.e. pieces of the diagram may be replaced by homotopy equivalent objects and morphisms). If one asks that they commute strictly, one is unlikely to have many interesting examples. On the other hand, if one asks that they commute up to homotopy, there may potentially be an enormous amount of data to specify, since there are an infinite number of diagrams to draw (e.g. for a group object A and for each n one must consider all the possible n-fold multiplication maps $A^n \to A$). There is an excellent discussion and resolution of parts of this problem in [May72].

The main idea used by Lurie in [Lur14], and which we will use here, is that for a quasicategory $\mathcal{C}$, a ring or group object A in $\mathcal{C}$ should be specified by an object $A^n \in \mathcal{C}^n$ for each n and symmetries relating all of the A^n. This is accomplished by having a larger quasicategory $\mathcal{C}^{\otimes}$ that looks something like all the $\mathcal{C}^n$ put together into one category, and having a quasicategory which indexes all these possible symmetries (called an ∞-operad) $\mathcal{O}^{\otimes}$ over which $\mathcal{C}^{\otimes}$ is fibered. We are of course being terribly brief here, but we will make these notions exact in the following section. First we will need to recall some basic facts about certain special fibrations of quasicategories and simplicial sets. These should be compared to the well known Grothendieck fibrations and opfibrations of topos theory.

2.2.1 Fibrations of Simplicial Sets

We recall some definitions from [Lur09] regarding (co)Cartesian fibrations of simplicial sets (in particular, quasicategories) and (co)Cartesian morphisms. The following is taken almost verbatim from Section 2.4 of [Lur09].

Definition 2.2.1.1 (Kan Fibration)**.** A morphism of simplicial sets $p : C \to D$ is a Kan fibration if for every horn inclusion $\Lambda^n_i \hookrightarrow \Delta^n$ and every commuting square like the following, there is a dotted arrow making the diagram commute:

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook, "\iota_i"'] & C \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & D \end{tikzcd}$$

Remark 2.2.1.2. Note that if C is a Kan complex as in Definition 2.1.1.10 then $C \to \Delta^0$ is a Kan fibration, so we might think of a Kan fibration $C \to D$ as a family of Kan complexes indexed by the simplicial set D.

Definition 2.2.1.3 (Inner Fibration)**.** A morphism of simplicial sets $p : C \to D$ is an inner fibration if for all $n \geq 0$, every inner horn inclusion $\Lambda^n_i \hookrightarrow \Delta^n$ for $0 < i < n$ and every commuting square like the following, there is a dotted arrow making the diagram commute:

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook, "\iota_i"'] & C \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & D \end{tikzcd}$$

Example 2.2.1.4. Note that if C is a quasicategory as in Definition 2.1.2.1 then $C \to \Delta^0$ is an inner fibration. Thus we may think of an inner fibration $C \to D$ as being a family of quasicategories indexed over D.

Definition 2.2.1.5 (Cartesian Morphism)**.** Let $f : x \to y$ be an edge of a simplicial set C and $p : C \to D$ an inner fibration of simplicial sets. Then we say that f is p-Cartesian if the induced functor

$$C_{/f} \to C_{/y} \times_{D_{/p(y)}} D_{/p(f)}$$

is a trivial Kan fibration. Say that f is p-coCartesian if f is p^{op}-Cartesian for $p^{op} : C^{op} \to D^{op}$.

Remark 2.2.1.6. The above definition may take some unraveling. Firstly, notice that there is a functor $C_{/y} \to D_{/p(y)}$ which takes objects in $C_{/y}$, i.e. edges $z \to y$, to $p(z) \to p(y)$ in $D_{/p(y)}$. There is also a functor $D_{/p(f)} \to D_{/p(y)}$ which simply takes an object of $D_{/p(f)}$, which can be represented by a commutative diagram like the following:

$$\begin{tikzcd} w \arrow[d] \arrow[dr] & \\ p(x) \arrow[r, "p(f)"] & p(y) \end{tikzcd}$$

to the edge $w \to p(y)$ in D. As such, we can form the pullback of simplicial sets $C_{/y} \times_{D_{/p(y)}} D_{/p(f)}$.

Intuitively, this is the category of pairs (g, ϕ) where $g : z \to y$ is an edge in C and ϕ is an edge in D as in the following diagram:

$$\begin{tikzcd} p(z) \arrow[d, "\phi"'] \arrow[dr, "p(g)"] & \\ p(x) \arrow[r, "p(f)"] & p(y). \end{tikzcd}$$

Now, note that $C_{/f}$ maps to this pullback by taking a cone over f:

$$\begin{tikzcd} z \arrow[d, "h"'] \arrow[dr, "g"] & \\ x \arrow[r, "f"] & y \end{tikzcd}$$

to the pair $(g, p(h))$. Demanding that the morphism $C_{/f} \to C_{/y} \times_{D_{/p(y)}} D_{/p(f)}$ is a trivial Kan fibration in particular means that it is an equivalence. Thus, the important point to take away from this definition is that if we have the data of a cone over $p(f)$ such that the edge over $p(y)$ comes from something in C, there's an essentially unique edge over x which, when applying p, recovers the given cone over $p(f)$.

Remark 2.2.1.7. Note that if there exists a Cartesian lift $f : x \to y$ in C of an edge $\phi : d \to d'$ in D then it is always essentially unique. Indeed, if we have two Cartesian lifts of $\phi : p(x) \to p(y)$, say f_1 and f_2, then we have a cone in D of the form

$$\begin{tikzcd} p(x) \arrow[d, "id_{p(x)}"'] \arrow[dr, "p(f_2)"] & \\ p(x) \arrow[r, "p(f_1)"] & p(y). \end{tikzcd}$$

So by the discussion in the previous remark there is an essentially unique cone in C lifting this cone:

$$\begin{tikzcd} x \arrow[d, "g"'] \arrow[dr, "f_2"] & \\ x \arrow[r, "f_1"] & y. \end{tikzcd}$$

Hence f_2 factors through f_1 and similarly f_1 factors through f_2. By an almost identical argument we can see that all lifts of ϕ factor through a Cartesian lift, if a Cartesian lift exists. For a more technical description see Remark 2.4.1.9 of [Lur09].

Definition 2.2.1.8 (Cartesian Fibration)**.** Let $p : C \to D$ be a morphism of simplicial sets. Then p is a Cartesian fibration if the following conditions are satisfied:

1. The morphism p is an inner fibration of simplicial sets.

2. For every edge $f : x \to y$ of D and every vertex $\tilde{y}$ such that $p(\tilde{y}) = y$ there exists a p-Cartesian edge $\tilde{f} : \tilde{x} \to \tilde{y}$ such that $p(\tilde{f}) = f$.

A morphism $p : C \to D$ of simplicial sets is a coCartesian fibration if $p^{op} : C^{op} \to D^{op}$ is a Cartesian fibration.

Remark 2.2.1.9. The importance of Cartesian fibrations lies in the fact that given a Cartesian fibration $p : C \to D$ and an edge $\phi : d \to d'$ in D, there is always a functorial way to pull back from the fiber over d' to the fiber over d. That is, given an object $y \in C$ such that $p(y) = d'$, there is an associated Cartesian morphism $f : x \to y$, which is telling us that $\phi^*(y) = x$. Similarly, for a coCartesian fibration we can always push forward functorially along edges in D. Making this precise here would require the further introduction of *locally Cartesian fibrations* which would be unproductive. We refer the interested reader to Remark 2.4.2.9 of [Lur09] and the theory preceding it therein.

2.2.2 ∞-Operads, Algebras and Modules

We briefly review some of the constructions and definitions of [Lur14] regarding multiplicative structures in quasicategories. For a more detailed discussion of ∞-operads and an extensive investigation of their properties, see Chapter 2 of [Lur14].

Definition 2.2.2.1. Define the quasicategory $\mathcal{F}in_*$ to be the nerve of the category of linearly ordered finite pointed sets with order preserving functions between them. We will denote the finite set $\{*, 1, \ldots, n\}$ by $\langle n \rangle$. We will say that a morphism $f : \langle n \rangle \to \langle m \rangle$ of $\mathcal{F}in_*$ is *inert* if for each element $k \in \{1, \ldots, m\} \subset \langle m \rangle$, $f^{-1}(k)$ has exactly one element.

Definition 2.2.2.2 (∞-Operads)**.** An ∞-operad is a functor between quasicategories $p : \mathcal{O}^{\otimes} \to \mathcal{F}in_*$ satisfying the following properties (where we write $\mathcal{O}^{\otimes}_{\langle n \rangle}$ to denote the fiber of p over $\langle n \rangle$ and ρ_i to denote the unique inert morphism $\langle n \rangle \to \langle 1 \rangle$ taking i to 1):

1. For every inert morphism $f : \langle n \rangle \to \langle m \rangle$ in $\mathcal{F}in_*$ and every object C in $\mathcal{O}^{\otimes}_{\langle n \rangle}$ there is a p-coCartesian morphism $\overline{f} : C \to C'$ lifting f. In particular, f induces a functor $f_! : \mathcal{O}^{\otimes}_{\langle n \rangle} \to \mathcal{O}^{\otimes}_{\langle m \rangle}$.

2. Let C and C' be objects of $\mathcal{O}^{\otimes}_{\langle n\rangle}$ and $\mathcal{O}^{\otimes}_{\langle m\rangle}$ respectively and let $f : \langle n\rangle \to \langle m\rangle$ be a morphism of $\mathcal{F}in_*$. Let $Map^{f}_{\mathcal{O}^{\otimes}}(C, C')$ be the union of those connected components of $Map_{\mathcal{O}^{\otimes}}(C, C')$ which lie over f. Choose p-coCartesian morphisms $C' \to C'_i$ lying over the inert morphisms $\rho^i : \langle n\rangle \to \langle 1\rangle$ for each $1 \leq i \leq n$. Then the induced map

$$Map^{f}_{\mathcal{O}^{\otimes}}(C, C') \to \prod_{1\leq i\leq n} Map^{\rho_i \circ f}_{\mathcal{O}^{\otimes}}(C, C'_i)$$

is a homotopy equivalence.

3. For every finite collection of objects $C_1, \ldots, C_n$ in $\mathcal{O}^{\otimes}_{\langle 1\rangle}$ there exists an object C in $\mathcal{O}^{\otimes}_{\langle n\rangle}$ and a collection of p-coCartesian morphisms $C \to C_i$ covering $\rho^i : \langle n\rangle \to \langle 1\rangle$.

Definition 2.2.2.3 (Inert Morphisms)**.** Let $p : \mathcal{O}^{\otimes} \to \mathcal{F}in_*$ be an ∞-operad. Then a morphism f in $\mathcal{O}^{\otimes}$ is inert if $p(f)$ is inert in $\mathcal{F}in_*$ and f is p-coCartesian.

Remark 2.2.2.4. Note that the data of an ∞-operad is *not* the same as the data of a coCartesian fibration of simplicial sets $p : \mathcal{O}^{\otimes} \to \mathcal{F}in_*$. That will be, following Definition 2.2.2.6, the data of a symmetric monoidal structure on $\mathcal{O}^{\otimes}_{\langle 1\rangle}$. In particular, a symmetric monoidal structure always defines an ∞-operad, but the converse is not true.

Definition 2.2.2.5 (Maps of ∞-operads)**.** Given two ∞-operads $p : \mathcal{O}^{\otimes} \to \mathcal{F}in_*$ and $p' : \mathcal{O}'^{\otimes} \to \mathcal{F}in_*$, a map of ∞-operads is a functor of quasicategories $\mathcal{O}^{\otimes} \to \mathcal{O}'^{\otimes}$ which preserves inert morphisms and causes the following triangle to commute in $sSet$:

$$\begin{tikzcd} \mathcal{O}^{\otimes} \arrow[r] \arrow[d, "p"] & \mathcal{O}'^{\otimes} \arrow[dl, "p'"] \\ \mathcal{F}in_* & \end{tikzcd} .$$

Definition 2.2.2.6. Let $\mathcal{C}^{\otimes}$ be a quasicategory, $p : \mathcal{O}^{\otimes} \to \mathcal{F}in_*$ an ∞-operad and $f : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ a coCartesian fibration of simplicial sets. If the composition $p \circ f : \mathcal{C}^{\otimes} \to \mathcal{F}in_*$ exhibits $\mathcal{C}^{\otimes}$ as an ∞-operad then we say that $\mathcal{C} = \mathcal{C}^{\otimes}_{\langle 1\rangle}$ is $\mathcal{O}$-monoidal and that f is a coCartesian fibration of ∞-operads. In particular, to say that $\mathcal{C}$ is a symmetric monoidal (or $\mathcal{F}in_*$-monoidal) quasicategory is to say that there is a coCartesian fibration of simplicial sets $\mathcal{C}^{\otimes} \to \mathcal{F}in_*$ such that $\mathcal{C} = \mathcal{C}^{\otimes}_{\langle 1\rangle}$.

Remark 2.2.2.7. Note that being a coCartesian fibration of ∞-operads is strictly stronger than being a coCartesian fibration of simplicial sets.

Definition 2.2.2.8 (Algebras for ∞-operads)**.** For a coCartesian fibration of ∞-operads $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$, we define the category of $\mathcal{O}$-algebras in $\mathcal{C}$ to be the quasicategory of ∞-operad maps (see Definition 2.1.2.7 of [Lur14]) from $\mathcal{O}^{\otimes}$ to $\mathcal{C}^{\otimes}$ which cause the following triangle to commute in $sSet$:

$$\begin{tikzcd} \mathcal{O}^{\otimes} \arrow[r] \arrow[dr, "id"'] & \mathcal{C}^{\otimes} \arrow[d, "p"] \\ & \mathcal{O}^{\otimes} \end{tikzcd}$$

This quasicategory will be denoted by $Alg_{\mathcal{O}}(\mathcal{C})$.

Definition 2.2.2.9 (Lax Monoidal Functors)**.** Given two $\mathcal{O}$-monoidal quasicategories $\mathcal{C}$ and $\mathcal{D}$, a *lax* $\mathcal{O}$*-monoidal functor* is a morphism of ∞-operads $C^{\otimes} \to D^{\otimes}$ such that the following triangle commutes in $sSet$:

$$\begin{tikzcd} \mathcal{C}^{\otimes} \arrow[r] \arrow[dr] & \mathcal{D}^{\otimes} \arrow[d, "p"] \\ & \mathcal{O}^{\otimes} \end{tikzcd}$$

Let $Alg_{\mathcal{C}/\mathcal{O}}(\mathcal{D})$ denote the quasicategory of lax $\mathcal{O}$-monoidal functors from $\mathcal{C}$ to $\mathcal{D}$.

Remark 2.2.2.10. Given Definitions 2.2.2.8 and 2.2.2.9, it is not hard to check that lax $\mathcal{O}$-monoidal functors carry $\mathcal{O}$-algebras to $\mathcal{O}$-algebras.

Definition 2.2.2.11. We recall some important ∞-operads that will be used throughout the paper:

1. The quasicategory $\mathcal{F}in_*$ itself is an ∞-operad with underlying quasicategory Δ^0. Let $\mathcal{C}$ be a quasicategory. We will refer to the objects of $Alg_{\mathcal{F}in_*}(\mathcal{C})$ as commutative algebra objects of $\mathcal{C}$ and often denote them by $CAlg(\mathcal{C})$.

2. Let $p : \mathcal{C}^{\otimes} \to \mathcal{F}in_*$ determine a symmetric monoidal structure on a quasicategory $\mathcal{C}$. Recall that there is an ∞-operad $\mathcal{A}ss^{\otimes} \to \mathcal{F}in_*$ which characterizes the structure of associative algebras. Then recall from Notation 4.1.1.9 of [Lur14] that the fiber product $\mathcal{C}^{\otimes} \times_{\mathcal{F}in_*} \mathcal{A}ss^{\otimes}$ admits a coCartesian fibration of ∞-operads over $\mathcal{A}ss^{\otimes}$ and as such it is a so-called planar operad (cf. Definition 4.1.1.6 of [Lur14]). We will refer to ∞-operad morphisms $\mathcal{A}ss^{\otimes} \to C^{\otimes} \times_{\mathcal{F}in_*} \mathcal{A}ss^{\otimes}$ as associative algebra objects of $\mathcal{C}$ and often denote them by $Alg(\mathcal{C})$.

3. Recall from Definition 5.1.0.2 of [Lur14] that we have the ∞-operads $\mathbb{E}_k^{\otimes}$ of "little k-cubes" which interpolate between $\mathcal{A}ss^{\otimes}$ and $\mathcal{F}in_*$. Indeed, $\mathbb{E}_1^{\otimes} \simeq \mathcal{A}ss^{\otimes}$, and there are canonical morphisms of ∞-operads $\mathbb{E}_k^{\otimes} \to \mathbb{E}_{k+1}^{\otimes}$ such that $\mathcal{F}in_* \simeq colim_k(\mathbb{E}_k^{\otimes})$. For a symmetric monoidal quasicategory $\mathcal{C}^{\otimes} \to \mathcal{F}in_*$, we define $\mathbb{E}_k$-algebras similarly to associative algebras, as sections

of the pullback fibration $\mathcal{C}^{\otimes} \times_{\mathcal{F}in_*} \mathbb{E}_k \to \mathbb{E}_k$ that preserve inert morphisms. We will refer to the objects of $Alg_{\mathbb{E}_k}(\mathcal{C})$ as $\mathbb{E}_k$-algebra objects of $\mathcal{C}$.

Remark 2.2.2.12. One should think of the above structure as yielding multiplications on $\mathcal{C}$ by giving ways of going between the fibers of p over $\langle n\rangle$ and $\langle m\rangle$, which are $\mathcal{C}^n$ and $\mathcal{C}^m$ respectively. Moreover, one should interpret the fact that the fibration is coCartesian as being a suitable quasi-categorical generalization of the notion from classical category theory of a Grothendieck opfibration. That is, it provides a mechanism for functorially pushing forward along paths in the base. An example of such a structure is for instance the categorical fibration over the category of affine varieties whose fiber over a variety $Spec(R)$ is the category of quasi-coherent sheaves on $Spec(R)$. This is an opfibration because one can take the direct image sheaf along a map of varieties $Spec(R) \to Spec(S)$.

Recall from Definition 3.3.3.8 of [Lur14] that we can also define modules over algebras with the language of ∞-operads.

Definition 2.2.2.13 (Operadic Modules)**.** From Definition 3.3.3.8 and Theorem 3.3.3.9 of [Lur14], we have that for a unital, coherent ∞-operad $\mathcal{O}^{\otimes}$, an $\mathcal{O}$-monoidal quasicategory $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ and an $\mathcal{O}$-algebra $A : \mathcal{O}^{\otimes} \to \mathcal{C}^{\otimes}$, we have an ∞-operad given by a coCartesian fibration $Mod_A^{\mathcal{O}}(\mathcal{C})^{\otimes} \to \mathcal{O}^{\otimes}$. Objects of the fiber of this fibration over $\mathcal{O}$ can be thought of as A-modules in $\mathcal{C}$ with a prescribed A-action. If the operad $\mathcal{O}^{\otimes}$ is clear, we will write simply $Mod_A(\mathcal{C})$ or Mod_A.

Remark 2.2.2.14. In the case that $n = 1$, $Mod_A^{\mathbb{E}_n}$ should be thought of as the quasicategory of bimodules over an associative algebra A. Moreover, for $n \geq 1$ and an $\mathbb{E}_n$-algebra R in a quasicategory $\mathcal{C}$, there is a forgetful functor $Mod_R^{\mathbb{E}_n}(\mathcal{C}) \to Mod_R^{\mathbb{E}_1}(\mathcal{C})$ associated to thinking of R as an $\mathbb{E}_1$-algebra (see Theorem 5.1.3.2 of [Lur14]). In this note we will primarily work with categories of left (or right) R-modules, typically denoted $LMod_R$. This choice is in line with the recent literature on Thom spectra, which we use heavily (e.g. [ABG+14], [ABG15] and [ACB14]).

Proposition 2.2.2.15 (Left Modules)**.** *Given an $\mathbb{E}_n$-monoidal quasicategory $\mathcal{C}$ and an $\mathbb{E}_n$-algebra R in $\mathcal{C}$, there is a category of left modules over R regarded as an $\mathbb{E}_1$-algebra, $LMod_R(\mathcal{C})$. This category is $\mathbb{E}_{n-1}$-monoidal.*

Proof. See Sections 4.2.1 and 5.1.4 of [Lur14]. □

Remark 2.2.2.16. Recall from Section 5.1.4 of [Lur14] that if $\mathcal{C}$ is symmetric monoidal and R is a $\mathcal{F}in_*$-algebra (i.e. an $\mathbb{E}_\infty$-algebra), then $LMod_R(\mathcal{C}) \simeq Mod_R^{\mathcal{F}in_*}(\mathcal{C})$. This equivalence is canonical.

2.3 Parameterized Homotopy Theory

Many of the examples in [Rog08] and [Rot09], and all of the examples we will produce in Chapter 4, are Thom spectra. In general, the spectra that behave the most like spaces are suspension spectra, i.e. spectra in the image of the functor $\Sigma^\infty_+ : \mathcal{T} \to \mathcal{S}$. Thom spectra can be thought of as one step away from suspension spectra, or as being "twisted" suspension spectra. More precisely, we can define a spectrum parameterized by a Kan complex X to be a map of simplicial sets $f : X \to \mathcal{S}$. Given such a parameterized spectrum, we can produce a single spectrum by taking the colimit (thought of as a functor of quasicategories). If the functor $f : X \to \mathcal{S}$ factors through the constant functor $* \to \mathcal{S}$ valued in $\mathbb{S}$, then the colimit of this object will be exactly $\mathbb{S}[X] = \Sigma^\infty_+ X$, the suspension spectrum of X. If, on the other hand, the functor f still takes every point of X to $\mathbb{S}$, but is not constant on the 1-simplices of X, we can think of any colimit of this functor as a *twisted* version of $\mathbb{S}[X]$. What's even more interesting is that if we have that $X \simeq BG$ for G some $\mathbb{E}_n$-monoidal Kan complex, and f takes values in $BGL_1(\mathbb{S})$, the sub-quasicategory of $\mathbb{S}$ generated by spectra equivalent to $\mathbb{S}$ and homotopy equivalences, then we can think of the functor $f : X \to \mathcal{S}$ as defining an action of G on $\mathbb{S}$. In this case, it's not hard to see that a colimit of f is in fact $\mathbb{S}/G$, the "homotopy quotient of $\mathbb{S}$ by G." We will use this intuition to great avail in Chapter 4, but right now only suggest it as motivation for describing the theory of parameterized spaces and quasicategories.

All of the theory of this section is taken from [ABG+14] and [ABG15], a pair of papers which have allowed a great deal of the technology of [Lur14] to be used in the study of Thom spectra and derived algebraic geometry.

2.3.1 Parameterized Spaces and Spectra

Of crucial importance to this section is a quasicategorical version of the classical Grothendieck correspondence between pseudofunctors and fibered categories (cf. [Vis08]) called *straightening and unstraightening* by Lurie in [Lur09]. The idea is that given a functor from a simplicial set to the quasicategory of quasicategories $f : D \to q\mathcal{C}at$ (what classically would be a stack or pseudofunctor), there should be a fibration of simplicial sets $p : C \to D$ such that for every vertex $d \in D$, we have $p^{-1}(d) \simeq f(d)$. Given f, we obtain p by gluing the $f(d)$ together for all the d in D (this is given explicitly as a certain kind of colimit in [GHN15]). Given a fibration $p : C \to D$, we obtain a functor $D \to q\mathcal{C}at$ by by defining $f(d) = p^{-1}(d)$.

In the same way that we may classically have a category fibered either in groupoids or in categories, there is a distinction for us between functors valued in Kan complexes (sometimes called

∞-groupoids) and functors valued in quasicategories (sometimes called ∞-categories). For the latter, we will see that for a simplicial set D there is an equivalence between $Fun(D, q\mathcal{C}at)$ and the quasicategory of Cartesian fibrations over D of Definition 2.2.1.8 which we will denote by $Cart(D)$. For the former, there is an equivalence between $Fun(D, \mathcal{T})$ and a certain quasicategory of fibrations over D called *left fibrations*, which we explain in Definition 2.3.1.4. All of this can be dualized to obtain equivalences $Fun(D^{op}, q\mathcal{C}at) \simeq coCart(D)$ and $Fun(D^{op}, \mathcal{T}) \simeq RFib(D)$, where $coCart(D)$ is the quasicategory of coCartesian fibrations over D, and $RFib(D)$ is the quasicategory of right fibrations over D. Obviously this description is scant on details. We refer the reader to Section 3.2 of [Lur09] for more on this. We will recall the important theorems here, as they will be useful in this and in future sections.

Remark 2.3.1.1. In general, though we will not often need this level of generality, for any quasicategory $\mathcal{C}$, we can define the quasicategory of $\mathcal{C}$-valued presheaves on a Kan complex K to be the quasicategory $Fun(K, \mathcal{C})$. The term "presheaf" must be taken with a grain of salt however. Typically when one thinks of presheaves on a topological space, one thinks of a contravariant functor from the category of open subsets of K (with inclusions as morphisms) into some other category, e.g. sets, rings, abelian groups or R-modules for some ring R. In the case of $Fun(K, \mathcal{C})$ however, the open sets of K as a topological space are effectively ignored by the functors, which attach an object of $\mathcal{C}$ to each 0-simplex of K, a morphism of $\mathcal{C}$ to each 1-simplex in K, etc. Appendix A.1 of [Lur14] makes clear that such objects should be thought of as locally constant (pre-)sheaves in the same way that one identifies fiber bundles with locally constant sheaves in classical topology.

Remark 2.3.1.2. Recall from 3.1.3 of [Lur09] that for a fixed simplicial set D there is a model category structure on the category of *marked simplicial sets over* D, denoted $sSet^+_{/D}$ which presents the quasicategory of Cartesian fibrations over D. This model category structure is called the *Cartesian model structure* and its simplicial nerve will be denoted $Cart(D)$. Similarly there is a *coCartesian model structure* on $sSet^+_{/D}$ whose simplicial nerve will be denoted $coCart(D)$. Indeed the 0-simplices of these quasicategories can be checked to be Cartesian and coCartesian fibrations of simplicial sets over D respectively (up to homotopy). This is proven in Proposition 3.1.4.1 of [Lur09].

The following theorem gives that a Cartesian fibration $C \to D$ of simplicial sets is indeed the same thing as a functor $D \to q\mathcal{C}at$:

Theorem 2.3.1.3. *There is an adjoint equivalence of quasicategories*

$$St : Cart(D) \overset{\sim}{\rightleftarrows} Fun(D^{op}, q\mathcal{C}at) : Un$$

between Cartesian fibrations over D and $q\mathcal{C}at$-valued presheaves on D^{op} called the straightening/unstraightening equivalence. Similarly, there is an equivalence between coCartesian fibrations over D and $q\mathcal{C}at$-valued presheaves on D.

Proof. The equivalence is obtained by applying the simplicial nerve functor to the Quillen equivalence given in Theorem 3.2.0.1 of [Lur09]. $\square$

The straightening/unstraightening equivalence can be refined in the case that we are only interested in functors valued in Kan complexes, but we must introduce the notion of a *left fibration* of simplicial sets first:

Definition 2.3.1.4. A morphism of simplicial sets $p : C \to D$ is a left fibration if for all $n \geq 0$, every horn inclusion $\Lambda^n_i \hookrightarrow \Delta^n$ for $0 \leq i < n$ and every commuting square like the following, there is a dotted arrow making the diagram commute:

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook, "\iota_i"] & C \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & D. \end{tikzcd}$$

One can define *right fibrations* by replacing the left hand vertical morphism in the above diagram with horn inclusions for $0 < i \leq n$.

Remark 2.3.1.5. From [Lur14] we have that for a simplicial set D there is a certain simplicially enriched model structure on $sSet_{/D}$ which we will not need to know the details of. This model category has left fibrations as its fibrant objects. We will call the underlying quasicategory of $sSet_{/D}$ with this model structure the *quasicategory of left fibrations over D* and denote it $LFib(D)$. Similarly there is a quasicategory whose fibrant objects are right fibrations, and we will denote its underlying quasicategory by $RFib(D)$. See Definition 2.1.4.5 of [Lur09] for more on this.

Theorem 2.3.1.6. *Let D be a simplicial set. Then there is an adjoint equivalence of quasicategories $RFib(D) \overset{\sim}{\rightleftarrows} Fun(D^{op}, \mathcal{T})$. Similarly, there is an equivalence between the quasicategory of left fibrations and $\mathcal{T}$-valued presheaves on D.*

Proof. This follows from applying the simplicial nerve to the Quillen equivalence of Theorem 2.2.1.2 of [Lur09]. See also Definition 1.6 of [MG15]. $\square$

Notation 2.3.1.7. Though it overloads the terminology, we will also call the adjoint functors of Theorem 2.3.1.6 the unstraightening and straightening functors, and denote them by $St : LFib(D) \rightleftarrows$

$Fun(D, \mathcal{T}) : Un$. Whether we mean the adjunction with respect to functors valued in quasicategories or functors valued in Kan complexes will be clear from context.

Lemma 2.3.1.8. *If K is a Kan complex then every left fibration $C \to K$ is a Kan fibration.*

Proof. See Proposition 2.1.3.3 of [Lur09]. □

Corollary 2.3.1.9. *Suppose K is a Kan complex. Then there is an equivalence of quasicategories between the quasicategory of Kan complexes over K, denoted $\mathcal{T}_{/K}$ and the quasicategory of functors $Fun(K, \mathcal{T})$.*

Proof. See Lemma 2.1.3.3 of [Lur09] or Lemma 5.16 of [ABG15] (with $\mathcal{X} = \mathcal{T}$). The main idea is that given a left fibration, it is also a Kan fibration (by the above lemma) and as such is represented by a vertex of the quasicategory $\mathcal{T}_{/K}$. □

Notation 2.3.1.10. We will also refer to the pair of adjoint equivalences

$$Un : \mathcal{T}_{/K} \rightleftarrows Fun(K, \mathcal{T}) : St$$

as the straightening and unstraightening adjunction, since $LFib(K) \simeq \mathcal{T}_{/K}$ when K is a Kan complex.

In other words, and not surprisingly, up to homotopy the data of a diagram of spaces in the shape of a simplicial set K is equivalent to the data of a fibration of spaces assembled over K. This fact will be useful later when we will be interested in moving between these two structures.

Definition 2.3.1.11 (Parameterized Spaces)**.** Let X be a Kan complex regarded as a quasicategory. Define the *quasicategory of spaces parameterized by X* to be the quasicategory of functors of quasicategories $Fun(X, \mathcal{T})$.

Example 2.3.1.12. Let $c_Y : X \to \mathcal{T}$ be the functor that takes each 0-simplex of X to a fixed space Y, every 1-simplex to the identity on Y and every higher simplex to the corresponding degenerate simplices on in $\mathcal{T}$ at Y. Equivalently, we can say that c_Y factors through the constant functor $* \to \mathcal{T}$ that picks out Y. We will call this functor the *constant functor valued in Y from X to $\mathcal{T}$*, leaving out X and $\mathcal{T}$ if they are clear from context. Note that the colimit of this functor in $\mathcal{T}$ is $Y \times X$. In particular this shows that every space X can be obtained as the colimit of a constant functor valued in the point $c_* : X \to \mathcal{T}$.

Remark 2.3.1.13. The quasicategory of spaces over a simplicial set X, denoted $\mathcal{T}_{/X}$, has one obvious symmetric monoidal structure given by taking the fiber product over X. This is the Cartesian product symmetric monoidal structure on $\mathcal{T}_{/X}$ coming from the Cartesian symmetric monoidal structure on $\mathcal{T}$. On the other hand, the functor category $Fun(X, \mathcal{T})$ has a Cartesian symmetric monoidal structure given by taking Cartesian product on the target. That is, given two functors $F, G : X \to \mathcal{T}$, we have $(F \times G)(x) \simeq F(x) \times G(x)$ for all $x \in X$. The unstraightening equivalence $Un : Fun(X, \mathcal{T}) \to \mathcal{T}_{/X}$ is a right adjoint, so product preserving, and as such is a symmetric monoidal equivalence. Thus it is clear that these two Cartesian symmetric monoidal structures correspond to each other across the straightening/unstraightening equivalence.

Remark 2.3.1.14 (Day Convolution)**.** Alternatively, if X is an $\mathbb{E}_n$-monoidal Kan complex, this quasicategory has a *different* monoidal structure which is $\mathbb{E}_n$-monoidal. This is the Day convolution product structure on the quasicategory $Fun(X, \mathcal{T})$. Indeed, we know from Corollary 6.13 of [ABG15] that whenever X is an $\mathbb{E}_n$-monoidal Kan complex the quasicategory of functors $Fun(X, \mathcal{T})$ is an $\mathbb{E}_n$-monoidal quasicategory. See Corollary 4.8.1.12 of [Lur14] or Section 6.2 of [ABG15] for more on this. We can then use the unstraightening/straightening equivalence to induce an $\mathbb{E}_n$-monoidal structure on $\mathcal{T}_{/X}$. This can intuitively be thought of as being the monoidal structure in which the tensor product of $Y \to X$ and $Z \to X$ is given by $Y \times Z \to X \times X \overset{\mu_X}{\to} X$, where μ_X is algebra structure map on X.

Definition 2.3.1.15 (Parameterized Spectra)**.** For a Kan complex X define the quasicategory of *spectra parameterized by* X to be the functor quasicategory $Fun(X, \mathcal{S})$.

Remark 2.3.1.16. An object of the quasicategory of spectra parameterized by X, or $\mathcal{S}$-valued presheaves on X, associates a spectrum to each 0-simplex of X, and so on. We may also sometimes refer to such objects as bundles of spectra over X although this terminology doesn't really make any sense. That is, there is no Kan complex Y living over X whose fibers can be identified with spectra. However the statements of Remark 2.3.1.14 hold mutatis mutandis for parameterized spectra. What's more, we have that the stabilization of the quasicategory of spaces over X, $Stab(Fun(X, \mathcal{T}))$, is equivalent to the quasicategory $Fun(X, Stab(\mathcal{T})) \simeq Fun(X, \mathcal{S})$. See Proposition 3.4 of [ABG15] for a rigorous statement and proof of this fact.

There is a large and interesting body of work regarding parameterized spectra and spaces, perhaps going all the way back to Waldhausen's work on G-equivariant homotopy theory (equivalently spaces over BG) [Wal85]. We would also be remiss not to mention the seminal work of May and Sigurdsson

[MS06]. In the language of quasicategories much work was done by Lurie in [Lur14] and later by Ando, Blumberg and Gepner in [ABG15]. We highly recommend all of these references, as we will not delve into this topic much further.

2.3.2 Thom Spectra

For a discrete commutative ring R, a principal $BGL_1(R)$-bundle over a space X "looks like" a family of 1-dimensional free R-modules parameterized by X. In the same vein, we can construct a space $BGL_1(R)$ when R is an $\mathbb{E}_n$-ring spectrum and use it to parameterize bundles of 1-dimensional free R-module spectra over X. The seminal quasicategorical work in this direction was completed in the celebrated "five author paper" of Ando, Blumberg, Gepner, Hopkins and Rezk [ABG+14] (for R an $\mathbb{E}_\infty$ or $\mathbb{E}_1$-ring) and later followed up on by Ando, Blumberg and Gepner for more general $\mathbb{E}_n$-ring spectra [ABG15]. We provide a bare-bones review of that material here in an effort to make later chapters comprehensible, but still strongly recommend the aforementioned papers.

Warning 2.3.2.1. To avoid writing it in every statement, we assume for the rest of this section that whenever we refer to an $\mathbb{E}_n$-algebra, n is always greater than 1. Without this assumption, $Pic(R)$ will not necessarily exist.

Definition 2.3.2.2 (Picard Space)**.** For R an $\mathbb{E}_n$-monoidal ring spectrum, define $Pic(R)$ to be the sub-quasicategory of $LMod_R$ spanned by R-modules which are invertible with respect to the tensor product over R and R-module equivalences as morphisms.

Theorem 2.3.2.3. *For an $\mathbb{E}_n$-ring spectrum R, $Pic(R)$ is a grouplike $\mathbb{E}_{n-1}$-monoidal Kan complex*

Proof. The fact that $Pic(R)$ is a Kan complex follows from taking only equivalences as morphisms. The fact that it is $\mathbb{E}_n$-monoidal is a result of Lemma 7.4 of [ABG15] and the fact that taking the largest Kan complex contained in a quasicategory is functorial and preserves products (since it is right adjoint to the inclusion $\mathcal{T} \hookrightarrow q\mathcal{C}at$). The fact that it is grouplike follows from the fact that every object has an inverse (up to homotopy). ☐

Definition 2.3.2.4. For R an $\mathbb{E}_n$-ring spectrum, let $BGL_1(R)$ denote the sub-quasicategory of $LMod_R$ spanned by R-modules which are equivalent to R and R-module morphisms as equivalences.

Remark 2.3.2.5. Since $Pic(R)$ is grouplike, it has a unit component. In this case, this component is the one containing R itself. It is not hard to see that this component is equal to $BGL_1(R)$.

Definition 2.3.2.6. Let $EGL_1(R)$ denote the full over-quasicategory $BGL_1(R)_{/R}$. Hence $EGL_1(R)$ is the quasicategory of one dimensional free R-modules with a *chosen* equivalence to R.

Because we've taken only invertible morphisms, the next lemma follows immediately.

Lemma 2.3.2.7. *The quasicategories $BGL_1(R)$ and $EGL_1(R)$ are Kan complexes.*

Lemma 2.3.2.8. *The Kan complex $EGL_1(R)$ is contractible.*

Proof. The objects (i.e. vertices, or 0-simplices) of $EGL_1(R)$ are homotopy equivalences of R-modules, $\phi : R' \to R$. The morphisms are boundaries of two simplices that can be filled in a homotopy coherent way:

$$\begin{tikzcd} R' \arrow[r] \arrow[dr] & R'' \arrow[d] \\ & R \end{tikzcd}$$

In other words, they are functors $\Delta^2 \to LMod_R$ which take the the edges and vertices of Δ^2 to the obvious thing. Suppose we have two morphisms in $EGL_1(R)$ between $\phi : R' \to R$ and $\psi : R'' \to R$. Then they are both determined up to a contractible space of choices by the composition $\psi^{-1} \circ \phi$. In other words, they must be equivalent. Hence for any object of $EGL_1(R)$, the space of morphisms to that object from any other object is contractible. As such, every object of $EGL_1(R)$ is final. Hence by Proposition 1.2.12.9 of [Lur09] (originally a result of Joyal), $EGL_1(R)$ is contractible. □

Remark 2.3.2.9. Suppose one has a morphism of $\mathbb{E}_n$-rings $\phi : R \to T$, then there is a functor $BGL_1(R) \to BGL_1(T)$ given by tensoring with T over R (where T is an R-algebra by ϕ). Similarly for $EGL_1(R) \to EGL_1(T)$.

Theorem 2.3.2.10. *For an $\mathbb{E}_n$-ring spectrum R there is a Kan fibration of $\mathbb{E}_{n-1}$-monoidal Kan complexes $EGL_1(R) \to BGL_1(R)$.*

Proof. The existence of the Kan fibration is given for the spaces by Corollary 1.14 of [ABG$^+$14]. The fact that it is a morphism of $\mathbb{E}_n$-spaces follows trivially from the fact that $EGL_1(R)$ is contractible. □

Definition 2.3.2.11 (Thom Spectra)**.** Let R an $\mathbb{E}_n$-ring spectrum and let X be a Kan complex equipped with a morphism of spaces $f : X \to BGL_1(R)$. Then we will define the Thom spectrum of f, denoted Mf, to be

$$colim(X \to BGL_1(R) \hookrightarrow Mod_R(\mathcal{S})).$$

Remark 2.3.2.12. If X is of the form BG for some $\mathbb{E}_1$-monoidal Kan complex G, then we can think of the morphism f defining an action of ΩX on R. For this reason, we will occasionally write

$R/\Omega X$ instead of Mf.

Definition 2.3.2.11 is taken from [ABG+14], but there is another definition we might use, taken from [ABG15]:

Remark 2.3.2.13 (Thom Spectra 2)**.** For an ∞-operad $\mathcal{O}^{\otimes}$ admitting a map from $\mathbb{E}_1^{\otimes}$, there is an adjoint pair $Fun(-,\mathcal{T}) : Alg_{\mathcal{O}}^{gp}(\mathcal{T}) \rightleftarrows Alg_{\mathcal{O}}(Pr^L) : Pic$ between grouplike $\mathcal{O}$-algebras in $\mathcal{T}$ and $\mathcal{O}$-algebras in Pr^L, the quasicategory of presentable quasicategories and colimit preserving functors between them. The left adjoint $Fun(-,\mathcal{T})$ takes a grouplike $\mathcal{O}$-monoidal Kan complex X to the ($\mathcal{O}$-monoidal) presentable quasicategory of presheaves on X, $Fun(X,\mathcal{T})$. The right adjoint takes a presentable $\mathcal{O}$-monoidal quasicategory $\mathcal{C}$ to $Pic(\mathcal{C})$, the sub-quasicategory of invertible $\mathcal{O}$-algebras and equivalences between them. Moreover if $\mathcal{C} = LMod_R$ for an $\mathbb{E}_n$-ring spectrum, and $\mathcal{O}^{\otimes} = \mathbb{E}_{n-1}^{\otimes}$, then $BGL_1(R)$ is the base point component of $Pic(LMod_R)$. There is a comonad associated to this adjunction, and for the case of $\mathcal{C} = LMod_R$, the counit $Fun(Pic(LMod_R),\mathcal{T}) \to LMod_R$ is called the *generalized Thom spectrum functor*. By inclusion of the base component, we may extend this to a functor $Fun(BGL_1(R),\mathcal{T}) \to LMod_R$.

Remark 2.3.2.14. Note that by the quasicategorical Grothendieck construction given in Definition 2.3.1.9, there is an equivalence between $Fun(BGL_1(R),\mathcal{T})$ and the overcategory $\mathcal{T}_{/BGL_1(R)}$. Thus this alternative construction takes as input the same type of data as Definition 2.3.2.11, and both have objects of $LMod_R$ as output.

Theorem 2.3.2.15. *Definition 2.3.2.11 and Remark 2.3.2.13 give equivalent functors.*

Proof. From Corollary 8.13 of [ABG+14] we have a unique characterization, up to equivalence of functors of quasicategories, of the Thom spectrum functor, and one checks that both functors described above satisfy that characterization. □

Definition 2.3.2.11 has the advantage of being easy to understand and even visualize: you take a diagram in $LMod_R$ in the shape of X, possibly twisted by some automorphisms, and you take its colimit. This has a natural interpretation in terms of performing fiber integration on a "bundle of spectra." On the other hand, we will see in Chapter 3 that the construction of Remark 2.3.2.13 is more useful for describing the coalgebraic structure on Thom spectra.

Notation 2.3.2.16. Let R, X and f be as in Definition 2.3.2.11. We will say that f defines the trivial R-line bundle on X if there exists a factorization of f as $X \to * \to BGL_1(R)$.

Proposition 2.3.2.17. *If $f : X \to BGL_1(R)$ defines the trivial R-line bundle on a space X then $Mf \simeq R[X] = R \otimes_{\mathbb{S}} \Sigma_+^{\infty} X$.*

Proof. From Corollary 8.1 of [ABG15], we know that the Thom spectrum functor factors in the following way: given a morphism of spaces $f : X \to BGL_1(R)$, we know that there is an object in the quasicategory $\mathcal{T}_{/BGL_1(R)}$ which is in the same homotopy class as f after passing to the homotopy category, call it f' (to be explicit, f' is the 1-simplex in $\mathcal{T}$ corresponding to the replacement of f by a Kan fibration and passing to the simplicial nerve); then we may, using the Grothendieck correspondence, take the corresponding functor $F' : X \to \mathcal{T}$ which takes $x \in X$ to $f'^{-1}(x)$; we then take suspension spectra fibrewise and apply the counit map of a certain adjunction (see Theorem 7.9 of [ABG15]) to obtain an object of $LMod_R$. The crucial fact is that since f (and thus f') is equivalent to a map $X \to * \to BGL_1(R)$, the associated functor F' is the one that takes the (homotopically) unique vertex of $BGL_1(R)$ to X and every morphism to the identity morphism. Then taking suspension spectra fibrewise yields the spectrum parameterized by $BGL_1(R)$ which has "fibers" equivalent to $\Sigma^\infty_+ X = \mathbb{S}[X]$. Since the counit functor $Fun(BGL_1(R), \mathcal{S}) \to LMod_R$ described above is strongly $\mathbb{E}_n$-monoidal (i.e. preserves monoidal units) and takes the constant $\mathbb{S}$-valued functor to R, it must take the functor valued in $\mathbb{S}[X]$ to $R[X]$. $\square$

If the map $X \to BGL_1(R)$ is a map of $\mathbb{E}_n$-monoidal Kan complexes then its colimit in $LMod_R$ is also an $\mathbb{E}_n$-algebra. This is codified by the following theorem (originally proven in slightly less generality by Lewis [Lew78]) of [ACB14] (also proven in [ABG15]):

Theorem 2.3.2.18. *If R is an $\mathbb{E}_{n+1}$-ring spectrum then there is a colimit preserving functor $\mathcal{T}_{BGL_1(R)} \to LMod_R$ which is $\mathbb{E}_n$-monoidal.*

Proof. See Theorem 2.8 of [ACB14] as well as Corollary 8.1 of [ABG15]. $\square$

3

Coalgebra

In the previous chapter we described algebraic structure in quasicategories, and in this chapter we will describe a completely new framework for *coalgebraic* structure in quasicategories. Coalgebraic structure in abelian groups (e.g. Hopf-algebras and corings) has played an important role in stable homotopy theory. For instance, many important spectral sequences, like the Adams-Novikov and Eilenberg-Moore spectral sequences, have derived cotensor products (or *cotor*) as their E_2-terms. And of course both of these spectral sequences have been absolutely essential to computations in stable homotopy theory. Additionally, the theory of coalgebras and corings is one basis for descent and Galois theory, as described in [BW03], which we will mimic later in our discussion of quasicategorical descent data and Hopf-Galois extensions of ring spectra.

In a more directly homotopy theoretic vein, recent work of Hess and Shipley has shown that a homotopical theory of comodules can be used to define the A-theory of topological spaces [HS14]. Hess has also defined a useful category of descent data as a certain simplicial model category of comodules [Hes10]. Though the cited work uses the theory of simplicial model categories, it was one of the main sources of inspiration for this thesis. We hope to recover some of the theory developed by Hess and Shipley, but start by just providing quasicategorical basics for coalgebras, bialgebras and comodules. In the final sections of this chapter, we give some obvious examples of this structure.

We take as our starting point the most naïve notion of $\mathcal{O}$-coalgebras in an $\mathcal{O}$-monoidal quasicategory $\mathcal{C}$: $\mathcal{O}$-algebras in the opposite category $\mathcal{C}^{op}$. However, because our monoidal structures are so complicated (as they remember all homotopy coherences), we must take some care in constructing the opposite $\mathcal{O}$-monoidal structure. Luckily the quasicategorical Grothendieck construction of Section 2.3 makes this entirely formal. The resulting coCartesian fibration of ∞-operads is described explicitly in [BGN14], but we will not need that level of detail in what follows.

3.1 Coalgebras, Bialgebras and Comodules

In this section we provide definitions of coalgebras, bialgebras and comodules in an arbitrary $\mathbb{E}_n$-monoidal quasicategory as well as $\mathcal{S}$, the quasicategory of spectra. We show in this section (Proposition 3.1.1.16) that all spaces are $\mathbb{E}_\infty$-coalgebras in $\mathcal{T}$ and that n-fold loop spaces stabilize to cocommutative $\mathbb{E}_n$-bialgebras in $\mathcal{S}$, as one would expect. These definitions and the resulting theory, except where otherwise noted, are a new addition to the literature on quasicategorical operadic structure.

3.1.1 Basic Definitions

An $\mathcal{O}$-monoidal structure on a quasicategory $\mathcal{C}$ induces an $\mathcal{O}$-monoidal structure on $\mathcal{C}^{op}$ which is unique up to a contractible space of choices. In particular, an $\mathcal{O}$-monoidal structure on $\mathcal{C}$, given by a coCartesian fibration $\mathcal{C}^\otimes \to \mathcal{O}^\otimes$, is the same data as a functor $\mathcal{C} : \mathcal{O}^\otimes \to q\mathcal{C}at$ (by the straightening and unstraightening correspondence of Theorem 2.3.1.3). We may compose this functor with the op-involution $op : q\mathcal{C}at \to q\mathcal{C}at$ that takes a quasicategory to its opposite. Then the composite functor $op \circ \mathcal{C} : \mathcal{O}^\otimes \to q\mathcal{C}at$ determines an $\mathcal{O}$-monoidal structure on $\mathcal{C}^{op}$. This is the content of Remark 2.4.2.7 of [Lur14]. Once we have such a functor, we may apply the unstraightening functor to obtain a coCartesian fibration $\widehat{\mathcal{C}}^\otimes \to \mathcal{O}^\otimes$ whose fiber over $\langle 1\rangle$ is $\mathcal{C}^{op}$. Note that this construction is entirely opaque. It does not give us any intuition for what $\widehat{\mathcal{C}}^\otimes$ looks like as a quasicategory beyond telling us that it describes $\mathcal{O}$-monoidal structure on $\mathcal{C}^{op}$. However, the interested reader is invited to [BGN14] in which an explicit quasicategory is described, denoted $(\mathcal{C}^{\otimes,\vee})^{op}$ admitting a coCartesian fibration of ∞-operads $(p^\vee)^{op} : (\mathcal{C}^{\otimes,\vee})^{op} \to \mathcal{O}^\otimes$ defining an $\mathcal{O}$-monoidal structure on $\mathcal{C}^{op}$. Therein it is shown that $(\mathcal{C}^{\otimes,\vee})^{op}$ is equivalent to our $\widehat{\mathcal{C}}^\otimes$.

Example 3.1.1.1. When $\mathcal{C}$ is a symmetric monoidal quasicategory whose monoidal structure is given by the categorical product, the induced symmetric monoidal structure on $\mathcal{C}^{op}$ is the one whose monoidal structure is given by the categorical coproduct. In particular one may take the category of Kan complexes $\mathcal{T}$, where the tensor product of two Kan complexes X and Y is exactly the Cartesian product $X \times Y$. Note that since every Kan complex admits a diagonal map $X \to X \times X$, every Kan complex is a coalgebra, and hence an algebra in $\mathcal{T}^{op}$. We make this precise in Proposition 3.1.1.16.

Note that if we can define an $\mathcal{O}$-monoidal structure on $\mathcal{C}^{op}$ then there exists a quasicategory of $\mathcal{O}$-algebras in $\mathcal{C}^{op}$ associated to this $\mathcal{O}$-monoidal structure, denoted $Alg_{\mathcal{O}}(\mathcal{C}^{op})$. However, the morphisms of this quasicategory are still those of $\mathcal{C}^{op}$. Hence $Alg_{\mathcal{O}}(C^{op})$ is the *opposite* of the quasicategory of $\mathcal{O}$-coalgebras in $\mathcal{C}$.

Definition 3.1.1.2 (Coalgebras)**.** Let $\mathcal{C}$ be an $\mathcal{O}$-monoidal quasicategory for $\mathcal{O}^{\otimes}$ an ∞-operad. Then define the quasicategory of $\mathcal{O}$-coalgebras in $\mathcal{C}$ to be $(Alg_{\mathcal{O}}(\mathcal{C}^{op}))^{op}$, which we will usually denote by $CoAlg_{\mathcal{O}}(\mathcal{C})$. If $\mathcal{O}^{\otimes} = \mathcal{F}in_*$, we will write $CCoAlg(\mathcal{C})$ for the quasicategory of cocommutative coalgebras in $\mathcal{C}$. If $\mathcal{O}^{\otimes} = \mathbb{E}_1^{\otimes}$ we will write $CoAlg(\mathcal{C})$ for the quasicategory of coassociative coalgebras in $\mathcal{C}$.

Warning 3.1.1.3. Beginning with an $\mathcal{O}$-monoidal quasicategory $\mathcal{C}$ with associated coCartesian fibration $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ we are producing a coCartesian fibration over $\mathcal{O}^{\otimes}$ describing the $\mathcal{O}$-monoidal structure on $\mathcal{C}^{op}$. However, the reader should be aware that the actual construction of this coCartesian fibration is highly non-trivial. As a whole, $(\mathcal{C}^{\otimes,\vee})^{op} \simeq \widehat{\mathcal{C}}^{\otimes}$ looks very different from $(\mathcal{C}^{\otimes})^{op}$.

For the next definition, recall from Proposition 3.2.4.3 and Variant 5.1.2.8 of [Lur14] (and subsequent discussion) that the quasicategory of $\mathbb{E}_k$-algebras in an $\mathbb{E}_{k+j}$-monoidal quasicategory is generally only $\mathbb{E}_j$-monoidal. As a result, if we are interested in discussing bialgebras in an $\mathbb{E}_n$-monoidal quasicategory, our constructions only allow us to work with bialgebras that have an $\mathbb{E}_j$-comonoidal structure and an $\mathbb{E}_k$-monoidal structure for $j, k \geq 0$ and $j + k = n$. We will call such bialgebras *co*$\mathbb{E}_j$-$\mathbb{E}_k$-bialgebras.

Definition 3.1.1.4 (*co*$\mathbb{E}_k$-$\mathbb{E}_j$-Bialgebras)**.** Let $\mathcal{C}$ be an $\mathbb{E}_n$-monoidal quasicategory. Then for any $k \leq n$ there is a quasicategory of $\mathbb{E}_k$-coalgebras in $\mathcal{C}$ (see Definition 3.1.1.2), $CoAlg_{\mathbb{E}_k}(\mathcal{C})$. As the opposite of a quasicategory of $\mathbb{E}_{n-k}$-algebras, $CoAlg_{\mathbb{E}_k}(\mathcal{C})$ is $\mathbb{E}_{n-k}$-monoidal. As such, for each $j \leq n-k$, there are quasicategories $Alg_{\mathbb{E}_j}(CoAlg_{\mathbb{E}_k}(\mathcal{C}))$. For a fixed $j, k < n$, we call $Alg_{\mathbb{E}_j}(CoAlg_{\mathbb{E}_k}(\mathcal{C}))$ the category of *co*$\mathbb{E}_k$-$\mathbb{E}_j$-bialgebras in $\mathcal{C}$. We will denote this category by ${}^kBiAlg_j(\mathcal{C})$ where the lower right index gives the degree of commutativity, and the upper left index gives the degree of cocommutativity.

Remark 3.1.1.5. In the above definition, if $n = \infty$, then we (informally) have that $n - k = n$ for every k. In other words, in a symmetric monoidal quasicategory, $Alg_{\mathbb{E}_k}$ is again symmetric monoidal (cf. Examples 3.2.4.4 of [Lur14]). As such, in a symmetric monoidal quasicategory, we can define ${}^mBiAlg_n$ for arbitrary m and n.

Remark 3.1.1.6. Note that for an $\mathbb{E}_{k+j}$-monoidal category $\mathcal{C}$, an object H of $Alg_{\mathbb{E}_k}(\mathcal{C}^{op})$ admits a lifting of the inclusion of the base point $\{*\} \to \langle 1\rangle$, inducing an algebra unit map $1_{\mathcal{C}} \to H$. Hence H

admits a counit $\varepsilon : H \to 1_{\mathcal{C}}$ in $CoAlg_{\mathbb{E}_k}(\mathcal{S})$. Similarly, H admits a comultiplication $\delta : H \to H \otimes H$ which is "$\mathbb{E}_k$-cocommutative" up to coherent higher homotopy. We have ensured that the $\mathbb{E}_j$-algebra structure on ${}^kBiAlg_j(\mathcal{C})$ is compatible with this coalgebra structure by demanding that this structure pulls back the $\mathbb{E}_j$-monoidal structure of $CoAlg_{\mathbb{E}_k}(\mathcal{S})$.

Remark 3.1.1.7. Recall that when defining an affine monoid scheme, one defines it to be a monoid object in the category of affine schemes. As a result, an affine monoid scheme is both a monoid and a comonoid, and more importantly, these two structures are compatible and satisfy certain Hopf-algebra-type diagrams. In other words, to produce a bialgebra, we either have an algebra whose structure maps are maps of coalgebras, or a coalgebra whose structure maps are maps of algebras. Both of these conditions will produce the necessary compatibility between these structures.

It remains an interesting question to try to develop an ∞-operad (or perhaps an ∞-PROP) whose algebras are bialgebras. There is work in this direction in the book of Hackney, Robertson and Yau [HRY15].

Lemma 3.1.1.8. *Let $\mathcal{C}$ be a symmetric monoidal quasicategory and let $\circledast$ denote the symmetric monoidal product on $CoAlg_{\mathbb{E}_k}(\mathcal{C})$ induced by the symmetric monoidal product on $CoAlg_{\mathbb{E}_k}(\mathcal{C})^{op} = Alg_{\mathbb{E}_k}(\mathcal{C}^{op})$ as the category of $\mathbb{E}_k$-algebras in $\mathcal{C}^{op}$. Then for $H, K \in CoAlg_{\mathbb{E}_k}(\mathcal{C})$, the underlying $\mathcal{C}$ object of $H \circledast K$ is equivalent to $H \otimes K$, where $\otimes$ denotes the symmetric monoidal product of $\mathcal{C}$.*

Proof. Let $\circledast^{op}$ be the symmetric monoidal structure on $Alg_{\mathbb{E}_k}(\mathcal{C}^{op})$. From Remark 3.2.4.4 of [Lur14] we recall that for each object J of $\mathcal{F}in_*$, there is a symmetric monoidal evaluation functor of ∞-operads $ev_J : CAlg(\mathcal{C}^{op})^{\otimes} \to (\mathcal{C}^{\otimes})^{op}$. In other words, if $\otimes^{op}$ is the symmetric monoidal structure on $\mathcal{C}^{op}$, $X \circledast^{op} Y \simeq X \otimes^{op} Y$. Since the *op*-involution preserves objects, this proves the Lemma. □

Corollary 3.1.1.9. *Let H be an object of ${}^kBiAlg_j(\mathcal{C})$ for $\mathcal{C}$ a symmetric monoidal quasicategory. Then the underlying object of H admits an $\mathbb{E}_j$-algebra structure.*

Proof. This follows from the fact that the evaluation functor given above is a symmetric monoidal functor (again, see Remark 3.2.4.4 of [Lur14]). □

Proposition 3.1.1.10. *Let $\mathcal{C}$ and $\mathcal{D}$ be small $\mathbb{E}_n$-monoidal quasicategories and $f : \mathcal{C} \to \mathcal{D}$ an $\mathbb{E}_n$-monoidal functor. Then for $j + k = n$, if H is an object of ${}^kBiAlg_j(\mathcal{C})$ then $f(H)$ is an object of ${}^kBiAlg_j(\mathcal{D})$.*

Proof. The statement that f is an $\mathbb{E}_n$-monoidal functor means in particular that f corresponds to a map of ∞-operads over $\mathbb{E}_n$ for which both of the vertical maps in the following diagram are coCartesian:

$$\begin{tikzcd}
\mathcal{C}^{\otimes} \arrow[r, "f^{\otimes}"] \arrow[dr] & \mathcal{D}^{\otimes} \arrow[d] \\
& \mathbb{E}_n
\end{tikzcd}$$

Equivalently, by Lurie's straightening formalism, there is a natural transformation of maps of ∞-operads in $Fun_{\mathcal{F}in_*}(\mathbb{E}_n, q\mathcal{C}at)$ from the functor representing the $\mathbb{E}_n$-structure on $\mathcal{C}$ to the functor representing the $\mathbb{E}_n$-structure on $\mathcal{D}$. This induces a functor $\tilde{f}$ in $Fun_{\mathcal{F}in_*}(\mathbb{E}_n \times \Delta^1, q\mathcal{C}at)$, which we can compose with $op : q\mathcal{C}at \to q\mathcal{C}at$ to produce another functor $\tilde{f}^{op}$ which is equivalent to f on objects. It follows formally that f preserves both monoidal and comonoidal structure. $\square$

Definition 3.1.1.11 (Comodules)**.** Let $\mathcal{C}$ be an $\mathbb{E}_n$-monoidal quasicategory and let H be an object of $Alg_{\mathbb{E}_k}(\mathcal{C}^{op})$ for $0 < k \leq n$. Then, using Proposition 2.2.2.15, we know that there is an $\mathbb{E}_{k-1}$-monoidal quasicategory $LMod_H(\mathcal{C}^{op})$. Hence we define the category of left comodules over H to be the quasicategory $LMod_H(C^{op})^{op}$. We will denote this category by $LComod_H(\mathcal{C})$ or $LComod_H$.

Lemma 3.1.1.12. *If $\mathcal{C}$ is an $\mathbb{E}_n$-monoidal category and A is an (at least) $\mathbb{E}_1$-coalgebra in $\mathcal{C}$ then the category $LComod_A(\mathcal{C})$ admits K indexed colimits for every small simplicial set K. Moreover, the forgetful functor $LComod_A(\mathcal{C}) \to \mathcal{C}$ preserves these colimits.*

Proof. One notices that the category of comodules is the opposite of a category of modules, which is is closed under limits as demonstrated in Corollary 4.2.3.3 of [Lur14] $\square$

Definition 3.1.1.13 (Cotensor Product)**.** Let $\mathcal{C}$ be an $\mathbb{E}_m$-monoidal quasicategory, let H be an object of $CoAlg_{\mathbb{E}_n}(\mathcal{C})$ for $0 < n \leq m$, and B and C be objects of $RComod_H$ and $LComod_H$ respectively. Then using Construction 4.4.2.7 of [Lur14] we can form a simplicial object $Bar_H(B, C)_\bullet$ in $\mathcal{C}^{op}$ called the two-sided bar construction of B and C over H. If the colimit of $Bar_H(B, C)_\bullet$ exists, we call it the relative tensor product of B and C over H, and sometimes denote it by $B \otimes_H C$. Let $Cobar_H(B, C)^\bullet$ denote the cosimplicial object of $\mathcal{C}$ corresponding to $Bar_H(B, C)_\bullet$. If the limit of $Cobar_H(B, C)^\bullet$ exists in $\mathcal{C}$ then we call it the cotensor product of B and C over H and denote it by $B \square_H C$.

Remark 3.1.1.14. Recall that the cosimplicial object defining the cotensor product of B and C over H can be visualized by the diagram:

$$B \otimes C \rightrightarrows B \otimes H \otimes C \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} B \otimes H \otimes H \otimes C \ldots$$

where the coface maps are given by the H-coaction on B, the diagonal map of H, and the unit map of H.

Example 3.1.1.15. Note that given a morphism of spaces $X \to Y$, X is a left and right Y-comodule by composing with the diagonal map on the right or left, e.g. the map $X \xrightarrow{\Delta} X \times X \to X \times Y$. Given two such maps $X \to Y$ and $Z \to Y$ then the cotensor product of X and Z over Y, $X \square_Y Z$, is simply the fibered product $X \times_Y Z$. This can of course be computed by using the Eilenberg-Moore spectral sequence and in general there is a Bousfield-Kan spectral sequence that allows us to try to compute the homotopy of a cotensor product. This is, in fact, the usual Bousfield-Kan spectral sequence associated to cosimplicial cobar construction.

We make some of this rigorous in the following two propositions that will not be a surprise to most algebraic topologists.

Proposition 3.1.1.16. *Any Kan complex X is a cocommutative coalgebra object of $\mathcal{T}$ equipped with the Cartesian symmetric monoidal structure.*

Proof. Recall that there is a coCartesian fibration $p : \mathcal{T}^{\otimes} \to \mathcal{F}in_*$ defining the Cartesian symmetric monoidal structure on $\mathcal{T}$ (the one given by taking Cartesian products of Kan complexes). Then the monoidal structure on $\mathcal{T}^{op}$ determined by $(p^{\vee})^{op} : (\mathcal{T}^{\otimes,\vee})^{op} \to \mathcal{F}in_*$ is the coCartesian monoidal structure given by the coproduct in $\mathcal{T}^{op}$. From Corollary 2.4.3.10 of [Lur14] we know that every object of $\mathcal{T}^{op}$ is a commutative algebra with respect to the coCartesian monoidal structure, with algebra structure coming from the universal property of coproducts. Equivalently, every space is a cocommutative coalgebra with respect to the product on $\mathcal{T}$, given explicitly by the diagonal map. $\square$

Corollary 3.1.1.17. *Let $f : X \to Y$ be a morphism in $\mathcal{T}$. Then X is a Y-comodule in the Cartesian symmetric monoidal structure on $\mathcal{T}$.*

Proof. We know that, using the Cartesian symmetric monoidal structure on $\mathcal{T}$, Y is a commutative algebra in $\mathcal{T}^{op}$. In $\mathcal{T}^{op}$, there is a morphism $f^{op} : Y \to X$ which is a morphism of commutative algebras (again, see Corollary 2.4.3.10 of [Lur14]). Hence X is a Y-algebra (and therefore a Y-module) in $\mathcal{T}^{op}$. As a result, X is clearly a Y-comodule in $\mathcal{T}$. On the level of points, the coaction is given by $x \mapsto (x, f(x))$. $\square$

Remark 3.1.1.18. Note that, as a result of Corollary 3.1.1.17, given any space X and a pointed space $* \to Y$, X supports a Y-comodule structure given by the zero map $X \to * \to Y$. We will call this the *trivial Y-comodule structure on X*.

Corollary 3.1.1.19. *If X is an $\mathbb{E}_n$-algebra in $\mathcal{T}$ then $\mathbb{S}[X]$, the suspension spectrum of X, is an object of ${}^{\infty}BiAlg_n(\mathcal{S})$.*

Proof. Recall from section 4.8 of [Lur14] that $\mathcal{T}$ is a commutative algebra in Pr^L, the category of presentable quasicategories, with monoidal structure given by the Cartesian product of spaces. Moreover, there is a symmetric monoidal functor $\mathbb{S}[-] = \Sigma^{\infty}_{+} : \mathcal{T} \to \mathcal{S}$ presenting $\mathcal{S}$ as a $\mathcal{T}$-algebra which takes the product of spaces to the smash product of suspension spectra. Thus there is, by virtue of the functoriality of the involution $\mathcal{C} \mapsto \mathcal{C}^{op}$ on $q\mathcal{C}at$, a symmetric monoidal functor $(\Sigma^{\infty}_{+})^{op} : \mathcal{T}^{op} \to \mathcal{S}^{op}$ which takes coalgebra objects in $\mathcal{T}$ to objects in $CCoAlg(\mathcal{S})$. In particular, similarly to the proof of Proposition 3.1.1.10, on n-fold loop spaces this functor can be lifted to ${}^{\infty}BiAlg_n(\mathcal{S})$, yielding the result for $\mathbb{S}[X]$. □

3.1.2 The Thom Diagonal is a Structured Coaction

We now wish to show that the Thom diagonal, as constructed in [ABG+14], is a structured comodule coaction. We refer the reader to [ABG+14], [ABG15] and [ACB14] or to our Section 2.3.2 for a recollection of the basic constructions. The main idea is that there is a cocommutative coaction in Kan complexes over $BGL_1(R)$ whose underlying map of Kan complexes is the diagonal map. We will essentially be taking this relevant structure in a simplicial model category and passing to quasicategories by way of an operadic nerve construction. As such we will need some preliminaries on model categories and discrete operads. The books of Hovey [Hov99] and Hirschhorn [Hir03] provide all the material on model categories that we will need. We refer the reader to [Lur14] for the operadic constructions we use, but excellent references can also be found in the introductory sections of [Her00], [BM07] and [Hor15].

Given a monoidal category C, the associated colored endomorphism operad will be denoted $End(C)$ (this is described by Variant 4.3.1.17 of [Lur14]). Given a colored operad O, the associated category of operators will be denoted by $O^{\otimes}$, and is given by Construction 2.1.1.7 of [Lur14]. If C is simplicially enriched then so will be $End(C)$ and $End(C)^{\otimes}$.

The proof of the following proposition describing a monoidal model structure on an overcategory was communicated to the author by Alexander Campbell, who attributed it to Ross Street.

Proposition 3.1.2.1. *Let $(M, \otimes, \mathbb{1})$ be a simplicial monoidal model category. If A in M is a monoid then the slice category $M_{/A}$ is also a simplicial monoidal model category and the forgetful functor $M_{/A} \to M$ is strictly monoidal.*

Proof. Given two objects $f : X \to A$ and $g : Y \to A$ in $M_{/A}$, we define their tensor product to be $X \otimes Y \stackrel{f \otimes g}{\to} A \otimes A \stackrel{\mu_A}{\to} A$, where μ_A is the monoid structure map of A. It is routine to check that this defines a monoidal structure on the ordinary category $M_{/A}$ and that the forgetful functor is strictly monoidal. Given two morphisms $f : X \to A$ and $g : Y \to A$, we define their internal mapping object in the following way: there is a morphism $\hat{\mu}_A : A \to Hom_M(A, A)$ which is adjoint to the multiplication $\mu_A : A \otimes A \to A$. Notice moreover that there is a morphism of mapping spaces $\lambda : Hom_M(X, Y) \to Hom_M(X, A)$. Thus we define the internal mapping space over A, denoted $Hom_{M_{/A}}((X, f), (Y, g)) \to A$, to be the following pullback:

$$\begin{tikzcd}
Hom_{M_{/A}}((X,f),(Y,g)) \arrow[rr] \arrow[d] & & Hom_M(X,Y) \arrow[d, "\lambda"] \\
A \arrow[r, "\hat{\mu}_A"] & Hom_M(A,A) \arrow[r] & Hom_M(X,A).
\end{tikzcd}$$

There is a model structure on $M_{/A}$ in which the fibrations, cofibrations and weak equivalences are precisely the morphisms which are such under the forgetful map, by Theorem 7.6.5 of [Hir03]. Finally, as an overcategory, it is routine to check that colimits in $M_{/A}$ are created in M by the forgetful functor, so as long as the pushout-product axiom is satisfied in M, it is also satisfied in $M_{/A}$. □

Example 3.1.2.2. In the above proposition we may take $M = sSet$, the category of simplicial sets, with the Quillen model structure and the standard simplicial enrichment (cf. Example 9.1.13 of [Hir03]). The monoidal structure is given by the Cartesian product of simplicial sets. If we take A to be a strict monoid, we obtain a simplicial monoidal model category structure on $sSet_{/A}$. Note that the monoidal structure thus obtained on $sSet_{/A}$ is *not* the Cartesian one.

Lemma 3.1.2.3. *Let A be a strict group object in sSet and let $sSet_{/A}$ have the simplicial monoidal model structure defined in Proposition 3.1.2.1 and Example 3.1.2.2. Then if $f : X \to A$ and $g : Y \to A$ are fibrant objects of $sSet_{/A}$ (equivalently fibrations in sSet), so is $\mu_A \circ (f \times g) : X \times Y \to A$.*

Proof. From Lemma 18.2 of [May67], we know that for any principal A-fibration of simplicial sets $A \stackrel{i}{\to} E \stackrel{p}{\to} B$, p is a Kan fibration. Since A acts principally on itself, the multiplication map $\mu_A : A \times A \to A$ is a principal A-fibration, and thus a Kan fibration. One can check from definitions that $X \times Y \stackrel{f \times g}{\to} A \times A$, is also a Kan fibration. Hence the composition $\mu_A \circ (f \times g)$ is a Kan fibration. Hence the tensor product in $sSet_{/A}$ preserves fibrant objects. □

Lemma 3.1.2.4. *Let C be a monoidal model category with full monoidal subcategory of bifibrant* *objects C°. Then the category of operators of the endomorphism operad of $(C^\circ)^{op}$, $End((C^\circ)^{op})^{\otimes}$, is equivalent to $(End(C^\circ)^{\otimes})^{op}$.*

Proof. An investigation of the relevant constructions in [Lur14] (or the other references given) makes the result clear. $\square$

Theorem 3.1.2.5. *Let X be a based Kan complex. Given a morphism $f : X \to BGL_1(R)$ for R an $\mathbb{E}_{n+1}$-ring spectrum, the associated Thom spectrum Mf is a comodule for the co$\mathbb{E}_n$-coalgebra $R[X]$.*

Proof. Let G be a model of $BGL_1(R)$ in the model category $sSet$ with the Quillen model structure which is a strict, associative topological group with a strict unit. Let $\widetilde{f} : \widetilde{X} \to G$ be a Kan fibration in $sSet$ which is equivalent to $f : X \to BGL_1(R)$ upon passing to the homotopy coherent nerve. It is not hard to check that $\widetilde{f} : \widetilde{X} \to G$ is a strict comodule for the trivial morphism $* : \widetilde{X} \to G$ that takes all of $\widetilde{X}$ to $1 \in G$, in $sSet_{/G}$ (with the overcategory model structure induced by the Quillen model structure on $sSet$). It is of course essential that G has a strict unit.

Also note that both $\widetilde{f} : \widetilde{X} \to G$ and $* : \widetilde{X} \to G$ are bifibrant objects in this model structure. They are cofibrant because all objects of $sSet$ with the Quillen model structure are cofibrant, and cofibrations in $sSet_{/G}$ are created by the forgetful map. They are fibrant because, by construction, their underlying maps to $\widetilde{X}$ are fibrations. Hence the triangle

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\Delta"] \arrow[dr, "\widetilde{f}"'] & \widetilde{X} \times \widetilde{X} \arrow[d, "\widetilde{f}\circ\pi_1"] \\ & G \end{tikzcd}$$

defines a coaction of $* : \widetilde{X} \to G$ on $\widetilde{f} : \widetilde{X} \to G$ in $(sSet_G)^\circ$, the full subcategory of bifibrant objects. In other words, we have a strict action of $* : \widetilde{X} \to G$ on $\widetilde{f} : \widetilde{X} \to G$ in $((sSet_{/G})^\circ)^{op}$. Applying Variant 4.1.3.17 of [Lur14] to $((sSet_{/G})^\circ)^{op}$ we see that we have a simplicial colored endomorphism operad $End((sSet_{/G})^\circ)^{op})$ encoding the opposite monoidal structure of $(sSet_{/G})^\circ$. Note that, since fibrancy is preserved by the tensor product of $(sSet_{/G})^\circ$, the mapping complexes of this colored operad are homotopy invariant, which is necessary if we wish to pass to the underlying monoidal quasicategory. The action described above induces a map of colored operads $\mathbf{LM} \to End(((sSet_{/G})^\circ)^{op}) \simeq End((sSet_{/G})^\circ)^{op}$ (where $\mathbf{LM}$ is the colored operad whose algebras are monoids and modules over them, defined in Definition 4.2.1.1 of [Lur14]). Taking operadic nerves, we obtain an algebra of the ∞-operad $\mathcal{LM}^{\otimes}$ in $N^{\otimes}_{\mathcal{A}ss}(sSet_{/G})^{op}$ (where the ∞-operad $\mathcal{LM}^{\otimes}$ is defined in 4.2.1.7 of [Lur14]).

Thus, we have shown that the diagonal map makes $f : X \to BGL_1(R)$ into an object of $LComod_{(X,*)}(\mathcal{T}_{BGL_1(R)})$. Since the Thom spectrum functor is strictly $\mathbb{E}_n$-monoidal, the result follows. $\square$

Remark 3.1.2.6. In general it is more difficult to take a coalgebra or comodule in a monoidal model category and produce a coalgebra or comodule in the underlying monoidal quasicategory. In this case we are able to leverage the fact that the fibrant objects in $sSet_{/G}$ are closed under the chosen monoidal structure. We're also relying on the fact that G is a strict topological group with a strict unit. In general, if one were to try to take the opposite category of the bifibrant objects in a simplicial monoidal model category, the mapping spaces of the associated colored operad would not be homotopy invariant.

3.1.3 Coalgebras From Comonads

We now describe how to obtain coalgebras from comonads. This is essentially an application of an Eilenberg-Watts type theorem, where we recognize comonads as coalgebras in endofunctor categories and produce coalgebras in the source category by evaluating at the generating object. This procedure makes recognizing categories of descent data as equivalent to comodule categories an essentially trivial exercise.

Definition 3.1.3.1. For any quasicategory $\mathcal{C}$ there is an $\mathbb{E}_1$-monoidal category of functors $Fun(\mathcal{C}, \mathcal{C})$, where the monoidal structure is given by composition (cf. Remark 4.7.2.31 of [Lur14]). If F is an object of $Fun(\mathcal{C}, \mathcal{C})$ then we say F is a comonad if F is an object of $CoAlg(Fun(\mathcal{C}, \mathcal{C}))$.

Theorem 3.1.3.2 (Eilenberg-Watts)**.** *Let B be an $\mathbb{E}_1$-algebra in a symmetric monoidal quasicategory $\mathcal{C}$, $Mod_B^{\mathbb{E}_1}$ the quasicategory of B-bimodules and $Fun^L(LMod_B^{\mathbb{E}_1}, LMod_B^{\mathbb{E}_1})$ the quasicategory of small colimit preserving endofunctors of the quasicategory of left B-modules. Then there is an equivalence of monoidal quasicategories $Mod_B^{\mathbb{E}_1} \xrightarrow{\sim} Fun^L(LMod_B^{\mathbb{E}_1}, LMod_B^{\mathbb{E}_1})$ given by $M \mapsto M \otimes_B -$. Its inverse is given by evaluation on B.*

Proof. See Proposition 7.1.2.4 of [Lur14]. $\square$

Corollary 3.1.3.3. *There is an equivalence of quasicategories between the quasicategory of colimit preserving comonads $F : LMod_B^{\mathbb{E}_1} \to LMod_B^{\mathbb{E}_1}$ and coalgebra objects of $Mod_B^{\mathbb{E}_1}$.*

Proof. This follows from the theorem by restricting the monoidal equivalence of Theorem 3.1.3.2 to quasicategories of algebras. $\square$

Corollary 3.1.3.4. *Let $\phi : A \to B$ be a morphism of (at least) $\mathbb{E}_1$-algebras of $\mathcal{C}$. Then the B-bimodule $B \otimes_A B$ is a coalgebra object of $Mod_B^{\mathbb{E}_1}$.*

Proof. Note that tensoring with $B \otimes_A B$ is equivalent to applying the forgetful functor $Mod_B^{\mathbb{E}_1} \to Mod_A^{\mathbb{E}_1}$ and then applying its left adjoint $- \otimes_A B : Mod_A^{\mathbb{E}_1} \to Mod_B^{\mathbb{E}_1}$. As the composition of a right adjoint followed by its left adjoint, this defines a comonad on $Mod_B^{\mathbb{E}_1}$. □

Remark 3.1.3.5. Note that one can obtain a more explicit construction of the coalgebra associated to a comonad by using the quasicategorical adjunction machinery of [RV]. In particular, there it is shown that an adjunction of quasicategories (in this case between $LMod_A^{\mathbb{E}_1}$ and $LMod_B^{\mathbb{E}_1}$) yields a cosimplicial object of $Fun(LMod_B^{\mathbb{E}_1}, LMod_B^{\mathbb{E}_1})$ satisfying the Segal condition, which determines, in light of Section 4.1.2 of [Lur14], a coassociative coalgebra object.

Theorem 3.1.3.6. *Let B be an $\mathbb{E}_n$-algebra of a symmetric monoidal quasicategory $\mathcal{C}$, for $n \geq 1$. Then $LMod_B^{\mathbb{E}_n}$ is an $\mathbb{E}_{n-1}$-monoidal category and the forgetful functor $Mod_B^{\mathbb{E}_n} \to LMod_B^{\mathbb{E}_n}$ is $\mathbb{E}_{n-1}$-monoidal.*

Proof. See Theorem 5.1.4.10 of [Lur14]. □

Corollary 3.1.3.7. *If $\phi : A \to B$ is a morphism of $\mathbb{E}_n$-algebra objects of $\mathcal{C}$ then $B \otimes_A B$ is an $\mathbb{E}_1$-coalgebra in $LMod_B^{\mathbb{E}_1}$.*

Remark 3.1.3.8. Often, given a monadic adjunction of categories $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ with F left-adjoint to G (i.e. one such that $\mathcal{D}$ is the category of algebras for the monad $G \circ F$), the category of comodules for the comonad $F \circ G$ is referred to as the category of descent data for this adjunction. It is a classical theorem that the category of descent data for the extension/restriction of scalars adjunction of a morphism of commutative rings $\phi : A \to B$ is equivalent to the category of comodules for the coring $B \otimes_A B$. This has been proven in the homotopical setting by Hess [Hes10], and we reprove her result for quasicategories here.

Theorem 3.1.3.9. *Given a morphism of $\mathbb{E}_n$-ring spectra $\phi : A \to B$, with associated comonad $F \in Fun^L(LMod_B^{\mathbb{E}_1}, LMod_B^{\mathbb{E}_1})$ there is an equivalence of quasicategories between $LComod_F(Fun^L(LMod_B^{\mathbb{E}_1}, LMod_B^{\mathbb{E}_1}))$ and $LComod_{B \otimes_A B}(LMod_B^{\mathbb{E}_1})$.*

Proof. The monoidal equivalence of Corollary 3.1.3.2 is lifted to an equivalence of module categories by the functor Θ of Section 4.8.3 of [Lur14]. Roughly, Θ takes as input a pair $(\mathcal{C}^{\otimes}, A)$ of a monoidal category $\mathcal{C}^{\otimes}$ and an object $A \in Alg_C$ and produces as output the pair of categories

$(\mathcal{C}^{\otimes}, Mod_A)$. An equivalence of quasicategories $f : \mathcal{C}^{\otimes} \xrightarrow{\sim} \mathcal{D}^{\otimes}$ which induces an equivalence of algebras $f(A) \simeq A'$ produces an equivalence $(\mathcal{C}^{\otimes}, Mod_A) \simeq (\mathcal{D}^{\otimes}, Mod_{A'})$. In this case we can apply Θ to $((Mod_B^{\mathbb{E}_1})^{op}, B \otimes_A B)$ which is equivalent to $(Fun^L(LMod_B^{\mathbb{E}_1}, LMod_B^{\mathbb{E}_1}), F)$. $\square$

4

Hopf-Galois Extensions

4.1 Discrete Hopf-Galois Theory

In this section we will review some of the theory of Hopf-Galois extensions of discrete rings. We do not recommend this section as a reference for learning this material. For instance, there are many applications of this material that we do not discuss here. We begin by reminding the reader of some basic facts about Galois extensions of rings. Much of the following is summarized from [Rog08].

4.1.1 Galois Extensions of Discrete Commutative Rings

Given an algebraic extension of fields $f : L \to K$ with $G = Aut_L(K)$, we say that f is Galois if $L \cong K^G$, where K^G is the subfield of K fixed by the action of G. Note that since G fixes $L \subset K$, there is always an inclusion map $L \to K^G$ even if f is not a Galois extension. This is the map that will witness the isomorphism $L \cong K^G$ if the extension is Galois. Recall also that if f is a Galois extension then the dimension of K as an L-vector space is equal to the cardinality of G. In other words, we can write $K \cong \prod_G L$ (this is the so-called normal basis theorem). From this isomorphism we see that as L-modules $K \otimes_L K \cong \prod_G K$. This final isomorphism is given by the map $(k_1 \otimes k_2) \mapsto \{k_1 g(k_2)\}_{g \in G}$. These observations lead to the definition of a Galois extension of rings:

Definition 4.1.1.1. Let $f : A \to B$ be a morphism of discrete commutative rings and G a group acting on B by A-algebra automorphisms. Then $f : A \to B$ is a Galois extension of commutative rings if

1. The canonical inclusion map $A \to B^G$ is an isomorphism.

2. The map $B \otimes_A B \to \prod_G B$ given by $b_1 \otimes b_2 \mapsto \{b_1 g(b_2)\}_{g \in G}$ is also an isomorphism.

Remark 4.1.1.2. It is not obvious that this is the *right* way to generalize the concept of a Galois extension of fields to commutative rings, but the description and examples given in [Gre92], as well as the homotopy theoretic examples of [Rog08], seem to indicate that this particular generalization is at the very least extremely useful.

Galois extensions of rings retain some of the nice properties of Galois extensions of fields:

Theorem 4.1.1.3. *Let $\phi : A \to B$ be a Galois extension of discrete commutative rings. Then B is a faithfully flat extension of A.*

Proof. See Lemma 1.9 of [Gre92]. □

Moreover, assuming certain easy-to-satisfy conditions on A and B, the fundamental theorem of Galois theory is satisfied by Galois extensions of commutative rings. For the following theorems we assume that A and B have no idempotents besides 0 and 1:

Theorem 4.1.1.4. *Let $f : A \to B$ be a G-Galois extension, $H \subset G$ a subgroup and $U = B^H$ the subalgebra of H-invariant elements. Then*

1. *$U \to B$ is an H-Galois extension of commutative rings.*

2. *If H is a normal subgroup of G then $A \to U$ is a G/H-Galois extension of A.*

Proof. See [Gre92] or [CHR65]. □

Remark 4.1.1.5. See also Appendix A of [Rot09] for an analogous statement in the case of Galois extensions of *non-commutative* rings.

Theorem 4.1.1.6. *Let $f : A \to B$ and G be as above. Assume that $U \subset B$ is a separable sub-A-algebra of B. Then there is a subgroup $H \subset G$ such that $U \cong B^H$.*

Proof. Again see [Gre92] or [CHR65]. □

Remark 4.1.1.7. Although we don't address it in this thesis, Galois extensions of rings were also generalized to spectra in [Rog08]. There it was shown that a large class of important ring morphisms in chromatic homotopy theory are Galois. Galois extensions of ring spectra should naturally embed into Hopf-Galois extensions, but we will not take the time to investigate such a functor.

4.1.2 Hopf-Galois Extensions of Discrete Rings

What is a Hopf-Galois extension?

The notion of a Hopf-Galois extension of rings further generalizes Galois extensions of rings by replacing the Galois group with a Galois Hopf-algebra (or sometimes just a bialgebra or augmented coalgebra). Instead of the Galois group acting on the extension over the base, the Hopf-algebra *coacts* on the extension over the base. First investigated by Kreimer and Takeuchi [KT81], the versatility of this idea has caused it to be used in many applications since. There is a direct analog of Galois descent in the theory of Hopf-Galois extensions. To be precise, for a G-Galois extension $\phi : A \to B$, recall that A can be recovered from B by taking G-fixed points. For an H-Hopf-Galois extension $\phi : A \to B$, which in particular gives B the structure of an H-comodule, A can be recovered from B by taking the *cofixed points*, or primitives, of the H-coaction on B. This descent procedure also generalizes from the category of B-modules with compatible G-action to the category of B-modules with compatible H-coaction.

In addition to generalizing Galois extensions of rings, Hopf-Galois extensions can also be seen as describing quotients of affine schemes by group actions. In other words, a map of rings $\phi : A \to B$ such that A can be recovered as the H-cofixed points of an H-coaction on B is roughly the same data as a map of affine schemes $Spec(\phi) : Spec(B) \to Spec(A)$ such that $Spec(A)$ can be recovered as the quotient of $Spec(B)$ by an action of $Spec(H)$. Yet another way to interpret this data is to say that $Spec(B)$ is a $Spec(H)$-torsor over $Spec(A)$, i.e. that $Spec(B) \times_{Spec(A)} Spec(B) \cong Spec(B) \times Spec(H)$. Recall that a $Spec(H)$ torsor over A is the same data as a principal $Spec(H)$ bundle over $Spec(A)$. Hence saying that the map $\phi : A \to B$ is an H-Hopf-Galois extension is another way to say that $Spec(B)$, as a cover of $Spec(A)$, locally looks like the affine group scheme $Spec(H)$. One of the benefits of formulating principal group scheme bundles in terms of Hopf-Galois extensions is that the latter does not require commutativity of the rings. Hence we can use Hopf-Galois extensions to talk about principal G-bundles of non-commutative schemes for G a so-called "quantum group" (cf. [BZ12]).

Definitions and Basic Properties

To make a rigorous definition of a Hopf-Galois extension of rings, we will first need to describe some auxiliary constructions:

Definition 4.1.2.1. Let B be a ring, H a bialgebra, and $c : B \to B \otimes H$ a ring map which determines a counital, coassociative coaction of H on B. Define the ring of H-cofixed points of B to be

$$B^{coH} = \{b \in B : c(b) = b \otimes 1\}.$$

Remark 4.1.2.2. Note that B^{coH} equalizes the diagram of rings $(c, 1_B \otimes u_H) : B \rightrightarrows B \otimes H$ constructed from the coaction and the unit map of H, and as such is a universal construction.

Definition 4.1.2.3. Given a morphism of rings $f : A \to B$ and a bialgebra H which coacts on B such that the coaction map $c : B \to B \otimes H$ is a map of A-algebras (where $B \otimes H$ has the A-module structure $a(b \otimes h) = ab \otimes h$), we define two maps:

- The *inclusion of cofixed points* map $i : A \hookrightarrow B^{coH}$ is the obvious inclusion induced by the fact that c is counital and a map of A-modules.
- The *torsor map* $\tau : B \otimes_A B \to B \otimes H$ is given by the composition $(\mu_B \otimes 1_H) \circ (1_B \otimes c)$, where μ_B is A-algebra structure map of B.

Definition 4.1.2.4 (Hopf-Galois Extension of Discrete Rings)**.** Let $f : A \to B$ be a morphism of discrete commutative rings, and a bialgebra H which coacts on B such that the coaction map $c : B \to B \otimes H$ is a map of A-algebras. We say that $f : A \to B$ is a Hopf-Galois extension if the maps i and τ of Definition 4.1.2.3 are bijections.

Example 4.1.2.5. Let $f : A \to B$ be a Galois extension of discrete commutative rings with finite Galois group G. Then $f : A \to B$ is a Hopf-Galois extension for the Hopf-algebra $Hom_{Mod_A}(A[G], A)$, where $A[G]$ is the group ring of G.

Theorems like 4.1.1.3, 4.1.1.4 and 4.1.1.6 are harder in general to come by for the case of Hopf-Galois extensions. But we do have the following:

Theorem 4.1.2.6. *Let $f : A \to B$ be a morphism of discrete commutative rings, and H a Hopf-algebra which coacts on B such that the coaction map $c : B \to B \otimes H$ is a map of A-algebras. Assume also that the antipode of H is bijective and that $A = B^{coH}$. Then the following are equivalent:*

1. *$f : A \to B$ is a Hopf-Galois extension with bialgebra H and B is a faithfully flat A-module.*
2. *The torsor map τ is surjective and B is an injective H-comodule.*

Proof. See Theorem 5.10 of [Mon09]. ☐

In other words, unlike the cases of Galois extensions of fields and of rings there are non-trivial conditions to be checked to ensure that Hopf-Galois extensions are of effective descent.

4.2 Hopf-Galois Extensions of Ring Spectra

In Chapter 3 we gave a brief definition of Thom spectra. That definition, and the following theory, did little to clarify *why* we're studying these objects or why their relationship with Hopf-Galois extensions might be interesting. We try to rectify this in the following impressionistic explanation.

4.2.1 Why Study Hopf-Galois Extensions in Homotopy Theory?

The motivation for studying Hopf-Galois extensions in this thesis is ultimately that they are intimately linked to the Thom spectra of stable homotopy theory. Thom spectra have played an important role in mathematics by manifesting a surprising link between homotopy theory and differential geometry. Their most immediate function is their relationship with bordism rings. In other words, given a map of classifying spaces $BG \to B\mathrm{O}$, we might ask if the classifying map of the stable normal bundle of a real manifold, $X \to BO$, lifts to form a commutative triangle

$$\begin{tikzcd} & BG \arrow[d] \\ X \arrow[ur] \arrow[r] & B\mathrm{O}. \end{tikzcd}$$

The collection of manifolds admitting such lifts, modulo the equivalence relation of identifying cobordant manifolds, forms a graded ring, which we denote MG_*. In particular, for the identity morphism $B\mathrm{O} \to B\mathrm{O}$, we have $M\mathrm{O}_*$, the ring of real manifolds modulo real bordisms, and for the block matrix inclusion $B\mathrm{U} \to B\mathrm{O}$ we obtain $M\mathrm{U}_*$, the complex bordism ring. We can relativize these rings by asking that each manifold X admit a map of topological spaces $X \to Y$ for some fixed topological space Y. Manifolds over Y, modulo bordism, also form a ring which we would denote by $MG_*(Y)$. As the notation suggests, the functors $MG_*(-) : \mathcal{T} \to GrRng$ from spaces to graded rings are a class of homology theories called *bordism theories*. They are represented by spectra MG called Thom spectra. As a result there are also associated *cobordism theories*, which are the cohomology theories $MG^*(-) : \mathcal{T}^{op} \to GrRng$ of Thom spectra. Thus Thom spectra, and their associated homology and cohomology theories, arise naturally in the process of trying to understand the structure of the bordism rings of differential geometry.

Our archetypical example for describing the surprising confluence of ideas around Thom spectra will be the complex cobordism spectrum $M\mathrm{U}$. The first thing to notice about $M\mathrm{U}$ is that $M\mathrm{U}^*(\mathbb{C}P^\infty) \cong M\mathrm{U}_*[[x_2]]$, i.e. the graded power series ring on a generator in degree 2. Since $\mathbb{C}P^\infty$ is a topological group, we can obtain a cogroup structure map $M\mathrm{U}_*[[x_2]] \to M\mathrm{U}_*[[x_2, y_2]] \cong$

$M\mathrm{U}_*[[x_2]]\widehat{\otimes}MU_*[[y_2]]$ defining a formal group law on $M\mathrm{U}_*$. Quillen showed in [Qui69] that this formal group law is in fact the *universal* formal group law. In other words, $M\mathrm{U}_*$ is the so-called Lazard ring and for any ring formal group law on a ring R, there is a map $MU_* \to R$ inducing it. This result is profound because it takes two areas of mathematics, differential geometry and formal algebraic geometry, and connects them in a non-obvious way. However it also has immediate computational consequences.

To wit, there is a spectral sequence, called the Adams-Novikov spectral sequence, whose input is the derived cotensor product of $M\mathrm{U}_*$ with itself over the Hopf-algebroid $M\mathrm{U}_*M\mathrm{U}$ (i.e. the homotopy groups of the cotensor product $M\mathrm{U}\square_{M\mathrm{U}\otimes M\mathrm{U}}M\mathrm{U}$ in spectra), and whose output is the stable homotopy groups of spheres, $\pi_*(\mathbb{S})$. By recognizing $M\mathrm{U}_*$ as the Lazard ring, we may now bring to bear information about formal algebraic geometry to computations of the stable homotopy groups of spheres (e.g. [Rav86]). However, by recognizing $M\mathrm{U}$ as a Hopf-Galois extension of the sphere spectrum (as Rognes did in [Rog08]), we can see the Adams-Novikov spectral sequence as a natural algebro-geometric construction. In particular, it is the descent spectral sequence which takes $\mathbb{S}[B\mathrm{U}]$-comodules to their *primitives*. When the comodule we start with is just $M\mathrm{U}$, its primitives are the sphere spectrum $\mathbb{S}$. This is a direct generalization of classical Galois descent computations. Given a G-Galois extension $\phi : A \to B$, we can take the fixed points of B-modules with G-action to obtain A-modules. Here, we've replaced G with a bialgebra, and fixed points with primitives (or *cofixed points*). Thus we see that the Adams-Novikov spectral sequence (which happens to exist, as a descent spectral sequence, for any Thom spectrum MG) is nothing more than the spectral analog of Galois cohomology. Hence the stable homotopy groups $\pi_*(\mathbb{S})$ can be interpreted as the k^{th} Galois cohomology of $\mathbb{S}[BU]$ with coefficients in in $M\mathrm{U}$. This is in line with Morava's philosophy of "getting behind the spectral sequence."

Moreover, following the discussion in Section 4.1.2, identifying maps of spectra as Hopf-Galois extensions gives them explicit geometric intuition. For instance, we show below that for a large class of $\mathbb{E}_n$-ring Thom spectra the unit morphism $\mathbb{S} \to MG$ is a $\mathbb{S}[BG]$-Hopf-Galois extension. Thus we may interpret this morphism as describing a principal $Spec(\mathbb{S}[BG])$-bundle in spectral affine varieties (though there doesn't exist a consensus on how to define such objects). This leads to a number of interesting questions that, as far as this author knows, are yet unanswered: What is a (flat) connection on such a principal bundle, and more importantly, what would the significance of such a thing be? How does the recognition of a morphism of ring spectra $\phi : A \to B$ as an H-Hopf-Galois extension affect the spectra $THH_A(B)$ and $TAQ_A(B)$? If $Spec(A)$ is to be thought of as a quotient of $Spec(B)$ by a spectral group scheme, how does this interact with Koszul duality, i.e. a correspondence between $Spec(H)$-modules and $BSpec(H)$-comodules?

Many of these ideas have already been mentioned or hinted at in Rognes' seminal manuscript in which he defines Hopf-Galois extensions of $\mathbb{E}_\infty$-ring spectra [Rog08]. Therein, Rognes uses the theory of ring spectra described in [EKMM95]. He shows that Galois extensions of commutative ring spectra play a central role in chromatic homotopy theory. In particular, the $K(n)$-local unit map $L_{K(n)}\mathbb{S} \to E_n$ of Morava E-theory is a (profinite) Galois extension with Galois group $\mathbb{G}_n$, the Morava stabilizer group. Rognes gave a natural generalization of Definition 4.1.2.4 and showed that, as mentioned above, the central morphism of chromatic homotopy theory, $\mathbb{S} \to M\mathrm{U}$, was a $\mathbb{S}[B\mathrm{U}]$-Hopf-Galois extension. The bialgebra mediating the descent from $M\mathrm{U}$ to $\mathbb{S}$ is $\mathbb{S}[B\mathrm{U}]$, i.e. the suspension spectrum of the classifying space of the infinite unitary group. Rognes explains that the torsor condition is precisely the classical Thom isomorphism $M\mathrm{U} \otimes_{\mathbb{S}} M\mathrm{U} \simeq M\mathrm{U} \otimes_{\mathbb{S}} B\mathrm{U}$ and that the cofixed points condition is given by the convergence of the Adams-Novikov spectral sequence.

Rognes' work was extended to associative (i.e. $\mathbb{E}_1$) ring spectra in the thesis of Roth [Rot09]. There, Roth showed that many Thom spectra besides $M\mathrm{U}$ are Hopf-Galois extensions of $\mathbb{S}$. The general idea is that given a map of loop spaces $f : X \to B\mathrm{U}$ the Thom spectrum Mf becomes a $\mathbb{S}[X]$-comodule by the Thom diagonal, and given certain relatively simple conditions on f, one can show that the cofixed points of this coaction recover $\mathbb{S}$. The first half of this statement is already proven in Section 3.1.2 above and since Thom spectra always admit Thom isomorphisms, we immediately have the torsor equivalence. The second half of this statement, regarding the cofixed points, is effectively a question about the Eilenberg-Moore spectral sequence when one is working with $H\mathbb{Z}$-oriented spectra. In other words, the cofixed points spectral sequence for homology is identical to the Eilenberg-Moore spectral sequence for the spaces determining the Thom spectra. As such, we get a number of cofixed points equivalences with very little effort.

4.2.2 Definitions and Basic Properties

Recall that if H is a cocommutative bialgebra in $\mathcal{S}$ (with an $\mathbb{E}_n$-algebra structure for some n), the quasicategory of H-comodules admits a symmetric monoidal structure. Moreover, since H is a bialgebra and admits a unit $\mathbb{S} \to H$, there is a *trivial comodule* functor $\mathcal{S} \to RComod_H$ which takes a spectrum A to the comodule with comodule structure map $A \simeq A \otimes \mathbb{S} \to A \otimes H$. We will say that an H-comodule *is a trivial H-comodule* if it is in the essential image of this functor.

Remark 4.2.2.1. Let H be a cocommutative $\mathbb{E}_n$-bialgebra in $\mathcal{S}$, and let B be an $\mathbb{E}_m$-algebra in $RComod_H$. Then the comodule structure map $c : B \to B \otimes H$ gives $B \otimes H$ the structure of an

$\mathbb{E}_k$-B-algebra, for $k = \min(n, m)$. On the other hand, if the existing coaction is non trivial, then the trivial coaction $B \simeq B \otimes \mathbb{S} \to B \otimes H$ induces a different B-algebra structure on $B \otimes H$.

Definition 4.2.2.2. Let A be an $\mathbb{E}_n$-ring spectrum and B be an $\mathbb{E}_m$-algebra in $LMod_A$ for $m < n$. Let H be a cocommutative $\mathbb{E}_n$-bialgebra in $\mathcal{S}$. Then we say that H *coacts on* B *over* A if B is an H-comodule and the following condition is satisfied:

- the coaction map $c : B \to B \otimes H$ induces an equivalence of A-algebras where $B \otimes H$ is equipped with the A-algebra structure induced by the composition of the unit map $A \to B$ with the *trivial* H-comodule structure map $B \simeq B \otimes \mathbb{S} \to B \otimes H$.

The following is an example of when this sort of coaction occurs:

Theorem 4.2.2.3. *Let R be an $\mathbb{E}_{n-1}$-ring spectrum and let $f : X \to BGL_1(R)$ be a morphism of $\mathbb{E}_n$-monoidal Kan complexes. Then $R[X]$ coacts on Mf over R.*

Proof. The coaction map of $R[X]$ on Mf is given by Thomifying the diagonal $\Delta : X \to X \times X$, thought of as a morphism over $BGL_1(R)$ (cf. Section 3.1.2). The trivial coaction is given by Thomifying the inclusion map $i_1 : X \hookrightarrow X \times X$. It's clear that the diagram $* \to X \rightrightarrows X \times X$ commutes (again, over $BGL_1(R)$). $\square$

Proposition 4.2.2.4. *Let H be a cocommutative $\mathbb{E}_n$-bialgebra in $\mathcal{S}$, and B an $\mathbb{E}_n$-algebra in $RComod_H$. Assume also that H-coacts on B over A, where B has the A-algebra structure induced by ϕ. Then the cotensor product has an A-algebra structure and receives a universal morphism $A \to B\square_H\mathbb{S}$.*

Proof. Since H-coacts on B over A, the cosimplicial construction $Cobar_H(B, \mathbb{S})$ whose totalization is $B\square_H\mathbb{S}$ lifts to a diagram of A-algebras and as such inherits a unit morphism $A \to B\square_H\mathbb{S}$. $\square$

The following definition, for $\mathbb{E}_\infty$-ring spectra in the symmetric monoidal simplicial model category of $\mathbb{S}$-modules, is due to Rognes [Rog08]. It was later generalized by Roth to $\mathbb{E}_1$-ring spectra, again using the category of $\mathbb{S}$-modules [Rot09]. Our definition below generalizes both of these definitions and only differs in that it is phrased in the language of quasicategories. The motivating example that should be kept in mind is the unit morphism $\mathbb{S} \to MU$ from the sphere spectrum to the complex cobordism spectrum.

Definition 4.2.2.5 (Hopf-Galois Extensions of Structured Ring Spectra)**.** Let H be a cocommutative $\mathbb{E}_n$-bialgebra in $\mathcal{S}$ and A an $\mathbb{E}_m$-algebra in $RComod_H$ that is a trivial H-comodule. Let B be an $\mathbb{E}_m$-algebra in $RComod_H$, $\phi : A \to B$ a morphism of $\mathbb{E}_m$-algebras and assume that H coacts on B over A. If:

1. the composite morphism $B \otimes_A B \overset{1\otimes c}{\to} B \otimes_A B \otimes H \overset{\mu_B \otimes 1}{\to} B \otimes H$ is an equivalence of $\mathbb{E}_n$-ring spectra, and

2. the cotensor product $B \Box_H \mathbb{S}$ exists and the canonical A-algebra map $A \to B \Box_H \mathbb{S}$ is an equivalence,

then we say that the map $\phi : A \to B$ is an *H-Hopf-Galois extension of $\mathbb{E}_m$-ring spectra.*

Example 4.2.2.6. There are a number of well known Thom spectra which are Hopf-Galois extensions of $\mathbb{S}$:

1. The complex cobordism unit map $\mathbb{S} \to M\mathrm{U}$ is a $\mathbb{S}[B\mathrm{U}]$-Hopf-Galois extension of $\mathbb{E}_\infty$-ring spectra (due to [Rog08]).

2. The unit map of mod-2 Eilenberg-MacLane spectrum $\mathbb{S} \to H\mathbb{Z}/2$ is a $\mathbb{S}[\Omega^2 S^3]$-Hopf-Galois extension of $\mathbb{E}_2$-ring spectra (due to Mahowald in [Mah79]).

3. The unit map $\mathbb{S} \to M\Xi$, as defined in [BR14], is a $\mathbb{S}[\Omega\Sigma\mathbb{C}P^\infty]$-Hopf-Galois extension of $\mathbb{E}_1$-ring spectra.

4. The unit map $\mathbb{S} \to X(n)$ of the $X(n)$ spectra defined by Ravenel [Rav86] is a $\mathbb{S}[\Omega\mathrm{SU}(n)]$-Hopf-Galois extension of $\mathbb{E}_2$-ring spectra.

5. The unit map to the n^{th} layer in the Postnikov tower of $M\mathrm{O}$, $\mathbb{S} \to M\mathrm{O}[n, \infty)$ is a $\mathbb{S}[B\mathrm{O}[n, \infty)]$-Hopf-Galois extension of $\mathbb{E}_\infty$-ring spectra for any of the n-connective covers of $B\mathrm{O}$, so long as $n > 0$ (e.g. $M\mathrm{SO}$, $M\mathrm{Spin}$, $M\mathrm{String}$).

Proofs that the above examples are indeed Hopf-Galois extensions follow immediately from the content of 4.2.3.4.

4.2.3 Intermediate Hopf-Galois Extensions of Thom Spectra

All of the Hopf-Galois extensions described by Rognes and Roth are of the form $\mathbb{S} \to MG$, for MG some Thom spectrum. The primary addition to the literature made by this thesis is the realization of a number of morphisms $MH \to MG$, for MH another Thom spectrum, as also being Hopf-Galois extensions. We show these extensions to be the sorts of extensions one might expect from a Galois correspondence. In other words we will require as input data a morphism of at least 2-fold loop spaces $H \to G$. We will then show that the extension $MH \to MG$ is a Hopf-Galois extension for the bialgebra $\mathbb{S}[B(G/H)]$. We also do not need to resort to case-by-case computations of complicated relative tensor products like $M\mathrm{String} \otimes_{M\mathrm{Spin}} M\mathrm{String}$. Using the notation above, our method allows us to recognize MG not just as a Hopf-Galois extension of MH, but actually as a Thom spectrum over MH (i.e. the colimit of a composition $B(G/H) \to BGL_1(MH) \to LMod_{MH}$). Thence we obtain Thom isomorphisms like $MG \otimes_{MH} MG \to MG \otimes \mathbb{S}[B(G/H)]$ (cf. Remark 4.2.3.16). In the $\mathbb{E}_\infty$ case, this has the additional benefit of giving MH as a quotient of MG by the action of some loop space and thus giving an alternative description of MG (cf. Remark 4.2.3.18). These constructions are entirely new, but should be compared to the work of Karpova for DGAs over a field [Kar14]. It also seems likely that an algebraist with significant tenacity could produce results of the kind we describe in Section 4.2.3 by computing with a Künneth spectral sequence (see Remark 4.2.3.17).

We should remark that our method of proving that certain spectra are Hopf-Galois extensions, even allowing for the differences between model categories and quasicategories, differs from that of Rognes and Roth. In particular, Rognes and Roth show that the Amitsur complex for the unit map of a Thom spectrum $\mathbb{S} \to MG$, which is a cosimplicial spectrum with $MG^{\otimes_{\mathbb{S}} n}$ in degree n, is equivalent to the cosimplicial spectrum $C^\bullet(MG, BG, \mathbb{S})$ which defines the cotensor product $MG \square_{\mathbb{S}[BG]} \mathbb{S}$ (cf. Definition 3.1.1.13). Then, using the fact that the totalization of the Amitsur complex is equivalent to $\mathbb{S}$ when MG is $H\mathbb{Z}$-oriented, they show that $MG \square_{\mathbb{S}[BG]} \mathbb{S} \simeq \mathbb{S}$. This equivalence of cosimplicial objects seems to be significantly harder to come by when one is working with quasicategories, so we determine the homotopy type of the relevant cotensor products in a different way. For an $\mathbb{S}[B(G/H)]$-Hopf-Galois extension $MH \to MG$, we show that there is a homology equivalence between the cotensor product $MG \square_{\mathbb{S}[B(G/H)]} \mathbb{S}$ and MH when MG is $H\mathbb{Z}$-oriented. The reason that this works is that the spectral sequence for $H\mathbb{Z} \otimes C^\bullet(MG, B(G/H), \mathbb{S})$ is isomorphic to the homology spectral sequence computing the homology of the cotensor product of *spaces* $BG \square_{B(G/H)} * = BG \times_{B(G/H)} *$. This last thing however, assuming that the associated Eilenberg-Moore spectral sequence has good properties, is precisely BH. In other words, we *untwist* the entire descent spectral sequence by smashing with $H\mathbb{Z}$, and then use the fact that cotensor products of spaces are precisely fibered products, as in Example 3.1.1.15.

Recall that the methods of [Rog08] and [Rot09] yield the following theorem:

Theorem. *Let X be a reduced and simply connected Kan complex. Let $f : X \to BGL_1(\mathbb{S})$ be a morphism of at least $\mathbb{E}_1$-spaces such that the composition $X \xrightarrow{f} BGL_1(\mathbb{S}) \to BGL_1(H\mathbb{Z})$ is nullhomotopic. Then the Thom spectrum of f, $Mf \simeq \mathbb{S}/\Omega X$ is a Hopf-Galois extension of $\mathbb{S}$ with Hopf-Galois bialgebra $\mathbb{S}[X]$.*

Our more general Theorem 4.2.3.4, which subsumes the theorem above, is the following:

Theorem. *Suppose $i : Y \to X$ and $f : X \to BGL_1(\mathbb{S})$ are morphisms of $\mathbb{E}_n$-monoidal Kan complexes for $n > 1$, with X and Y reduced and simply connected. If the composition $X \xrightarrow{f} BGL_1(\mathbb{S}) \to BGL_1(H\mathbb{Z})$ is nullhomotopic then there is a triangle of Hopf-Galois extensions of $\mathbb{E}_{n-1}$-monoidal ring spectra, where the associated bialgebras are written over their respective extensions:*

$$\begin{tikzcd}
\mathbb{S} \arrow[rr, "{\mathbb{S}[X]}"] \arrow[dr, "{\mathbb{S}[Y]}"'] & & Mf \\
& M(f \circ i) \arrow[ur, "{\mathbb{S}[X/Y]}"'] &
\end{tikzcd}$$

Warning 4.2.3.1. There is some inconsistency in our notation which depends on the situation. For instance, $M\mathrm{U}$ is obtained from the complex j-homomorphism $BU \to BGL_1(\mathbb{S})$, so by our usual notation should be denoted Mj, i.e. the Thom spectrum depends on the map, not just the underlying space. However, classically, for any group G admitting a map $f : G \to \mathrm{U}$ (or $G \to O$), the Thom spectrum associated to $BG \xrightarrow{f} B\mathrm{U} \xrightarrow{j} BGL_1(\mathbb{S})$ was denoted by MG rather than Mf. Even more confusingly, given a map of spaces $X \to BGL_1(R)$, we can consider the induced map of loop spaces $\Omega X \to GL_1(R)$. This latter map is the data of an action of ΩX on R. Hence we might write the colimit of this action (given by the colimit of the composite $X \to BGL_1(R) \to LMod_R$) by $R/\Omega X$. We find that this latter formulation is the most intuitively helpful for understanding our Theorem 4.2.3.4, which is effectively about taking iterated structured quotients of ring spectra by $\mathbb{E}_n$-actions. However, since the notation Mf is more familiar to most readers, we will try to use both as often as possible. The only time we will write MX or MG will be in the case of non-technical exposition or when we are referring to well known classical Thom spectrum like $M\mathrm{U}$ or $M\mathrm{String}$.

Homogeneous Spaces of $\mathbb{E}_n$-monoidal Kan Complexes

We begin by briefly investigating the notion of the homogeneous space associated to a map of $\mathbb{E}_n$-spaces. This concept should be compared to the homogeneous space obtained by quotienting a Lie group G by an inclusion $H \hookrightarrow G$ of Lie groups.

Definition 4.2.3.2 (Quotients of $\mathbb{E}_n$-spaces)**.** Let $i : Y \to X$ be a morphism of $\mathbb{E}_n$-monoidal Kan complexes. Define the quotient of X by Y, denoted X/Y, to be the relative tensor product $X \otimes_Y *$ in the sense of 4.4.2 of [Lur14], where the Y-module structure on X is determined by i. Recall that as the cone point of an operadic colimit diagram X/Y admits a universal quotient morphism $q : X \to X/Y$. Note that i is not made explicit in the notation, but will always be clear from context.

Lemma 4.2.3.3. *If $i : X \to Y$ is a morphism of $\mathbb{E}_n$-monoidal Kan complexes then X/Y is an $\mathbb{E}_{n-1}$-monoidal Kan complex and the quotient morphism $q : X \to X/Y$ is a morphism of $\mathbb{E}_{n-1}$-monoidal Kan complexes.*

Proof. To construct X/Y as an $\mathbb{E}_{n-1}$-monoidal Kan complex, we first notice that the relative bar construction of Section 4.4 of [Lur14] is constructed from the data of modules over an $\mathbb{E}_1$-algebra and that $Alg_{\mathbb{E}_n}(\mathcal{T}) \simeq Alg_{\mathbb{E}_1}(Alg_{\mathbb{E}_{n-1}}(\mathcal{T}))$. In other words the operadic bar construction of Construction 4.4.2.7 of [Lur14] is relative to the operad $\mathcal{O}^{\otimes} \simeq \mathbb{E}_{n-1}^{\otimes}$ and the bar construction $Bar_{\bullet}(X, Y, *)$ is computed in the quasicategory $Alg_{\mathbb{E}_{n-1}}(\mathcal{T})$. As the forgetful functor $Alg_{\mathbb{E}_{n-1}}(\mathcal{T}) \to \mathcal{T}$ is $\mathbb{E}_{n-1}$-monoidal it preserves the relevant structure on $X/Y = X \otimes_Y *$ and on the map it receives from $X \times * \simeq X$. $\square$

The Main Theorem

Theorem 4.2.3.4. *Suppose $i : Y \to X$ and $f : X \to BGL_1(\mathbb{S})$ are morphisms of $\mathbb{E}_n$-monoidal Kan complexes for $n > 1$, with X and Y reduced and simply connected. If the composition $X \xrightarrow{f} BGL_1(\mathbb{S}) \to BGL_1(H\mathbb{Z})$ is nullhomotopic then there is a triangle of Hopf Galois extensions of $\mathbb{E}_{n-1}$-monoidal ring spectra, where the associated bialgebras are written over their respective extensions:*

$$\begin{tikzcd} \mathbb{S} \arrow[rr, "{\mathbb{S}[X]}"] \arrow[dr, "{\mathbb{S}[Y]}"'] & & Mf \\ & M(f \circ i) \arrow[ur, "{\mathbb{S}[X/Y]}"'] & \end{tikzcd}$$

We will prove this theorem using two propositions (which in turn will rely on a number of lemmas). It should be clear from the results of [ABG+14] [ABG15] and [ACB14] that $\mathbb{S} \to Mf$ and $\mathbb{S} \to M(f \circ i)$ are both Hopf-Galois extensions with bialgebras $\mathbb{S}[X]$ and $\mathbb{S}[Y]$ respectively (though both of these statements follow from the content of the following proof). Thus it remains to show that the morphism $M(f \circ i) \to Mf$ is a Hopf-Galois extension with bialgebra $\mathbb{S}[X/Y]$. We will show that Mf can be produced as a Thom spectrum over $M(f \circ i)$, which will immediately yield the coaction of $\mathbb{S}[X/Y]$ on Mf over $M(f \circ i)$ as well as the torsor equivalence $Mf \otimes_{M(f \circ i)} Mf \simeq Mf \otimes \mathbb{S}[X/Y]$ (as the Thom diagonal and Thom isomorphism respectively). Then we will show that $Mf \square_{\mathbb{S}[X/Y]} \mathbb{S} \simeq M(f \circ i)$.

Proposition 4.2.3.5. *There is a morphism of $\mathbb{E}_{n-1}$-monoidal Kan complexes $X/Y \to BGL_1(M(f \circ$*

i)) such that the colimit of the composite morphism $X/Y \to BGL_1(M(f\circ i)) \to LMod_{M(f\circ i)}$ is equivalent to Mf.

Proof. Let X/Y denote an $\mathbb{E}_{n-1}$-monoidal quotient Kan complex of X by the Y-action on X induced by i following Definition 4.2.3.2. By Lemma 4.2.3.9, the fiber of the universal morphism $X \to X/Y$ is an $\mathbb{E}_{n-1}$-monoidal Kan complex which is equivalent to $Y \xrightarrow{i} X$. Hence by Lemma 4.2.3.10 the $\mathbb{E}_{n-1}$-monoidal left Kan extension of $X \xrightarrow{f} BGL_1(\mathbb{S}) \hookrightarrow \mathcal{S}$ along $q : X \to X/Y$ takes the unique 0-simplex of X/Y to $M(f\circ i)$. By Proposition 4.2.3.12, this Kan extension factors as a morphism of $\mathbb{E}_{n-1}$-monoidal Kan complexes through $BGL_1(M(f\circ i))$. Taking the Thom spectrum of the induced morphism $X/Y \to BGL_1(M(f\circ i))$ produces $M(f\circ i)/(\Omega(X/Y))$ as a Thom spectrum over $M(f\circ i)$. By Lemma 4.2.3.11 and Corollary 3.1.4.2 of [Lur14] we have that the left operadic Kan extension along $X \to X/Y$ followed by the left operadic Kan extension along $X/Y \to *$ is equivalent to the left operadic Kan extension along $X \to *$ (i.e. Kan extensions compose). Thus the iterated Kan extension which produces $M(f\circ i) = \mathbb{S}/\Omega Y$ and then quotients it by the action of $\Omega(X/Y)$ is equivalent to the one-step Kan extension $\mathbb{S}/\Omega X \simeq Mf$. Hence Mf is produced as a Thom spectrum over $M(f\circ i)$. This fact alone gives us a coaction $Mf \to Mf \otimes \mathbb{S}[X/Y]$ and the torsor condition

$$Mf \otimes_{M(f\circ i)} Mf \simeq Mf \otimes \mathbb{S}[X/Y].$$

□

Proposition 4.2.3.6. *The $\mathbb{S}[X/Y]$ cofixed point spectrum of Mf, computed by the cotensor product $Mf\square_{\mathbb{S}[X/Y]}\mathbb{S}$, where $\mathbb{S}$ has the trivial $\mathbb{S}[X/Y]$-coaction, is equivalent to $M(f\circ i)$.*

Proof. For the following proof we work within $\mathcal{T}_{/BGL_1(\mathbb{S})}$ equipped with the Day convolution symmetric monoidal structure. We will sometimes denote an object of $\mathcal{T}_{BGL_1(\mathbb{S})}$ by (X, f) instead of $f : X \to BGL_1(\mathbb{S})$. We start with the morphism of $\mathbb{E}_n$-monoidal Kan complexes $f : X \to BGL_1(\mathbb{S})$. Note that by an argument identical to the proof of Theorem 3.1.2.5, the quotient map $X \to X/Y$ induces a cocommutative coaction on $f : X \to BGL_1(\mathbb{S})$ in $\mathcal{T}_{BGL_1(\mathbb{S})}$ by the coalgebra $* : X/Y \to BGL_1(\mathbb{S})$ (i.e. the trivial map from X/Y to $BGL_1(\mathbb{S})$). Moreover, since X/Y is a Kan complex, and thus a unital coalgebra in $\mathcal{T}$, $* : X/Y \to BGL_1(\mathbb{S})$ is a unital coalgebra in $\mathcal{T}_{BGL_1(\mathbb{S})}$, hence there is a coaction of $* : X/Y \to BGL_1(\mathbb{S})$ on the point $* \to BGL_1(\mathbb{S})$. We then take the cosimplicial cobar construction in $\mathcal{T}_{BGL_1(\mathbb{S})}$, $Cobar^\bullet((X, f), (X/Y, *), (*, *))$. By tensoring up with $H\mathbb{Z}$ there is a morphism of $\mathbb{E}_\infty$-monoidal Kan complexes $BGL_1(\mathbb{S}) \to BGL_1(H\mathbb{Z})$. Thus by composition we extend our cobar construction to a cosimplicial object in $\mathcal{T}_{BGL_1(H\mathbb{Z})}$. Recall, however, that

$f : X \to BGL_1(\mathbb{S}) \to BGL_1(H\mathbb{Z})$ is null, thus there is an equivalence of cosimplicial objects over $BGL_1(H\mathbb{Z})$ between the one with bottom map $X \xrightarrow{f} BGL_1(\mathbb{S}) \to BGL_1(H\mathbb{Z})$ and the one with bottom map $X \xrightarrow{*} BGL_1(H\mathbb{Z})$. As the Thom spectrum functor $BGL_1(H\mathbb{Z}) \to LMod_{H\mathbb{Z}}$ is symmetric monoidal, we have an equivalence of cosimplicial $H\mathbb{Z}$-modules $H\mathbb{Z} \otimes Cobar^{\bullet}(Mf, \mathbb{S}[X/Y], \mathbb{S})$ and $H\mathbb{Z} \otimes Cobar^{\bullet}(\mathbb{S}[X], \mathbb{S}[X/Y], \mathbb{S})$. Thus the Bousfield-Kan spectral sequences for each of these objects are isomorphic. The Bousfield-Kan spectral sequence for the latter object is precisely the Eilenberg-Moore spectral sequence computing the integral homology of Y, and converges strongly since X/Y is simply connected (see 4.1 of [Bou87]). Hence the unit map of the former, $H\mathbb{Z} \otimes M(f \circ i) \to H\mathbb{Z} \otimes (Mf \square_{\mathbb{S}[X/Y]} \mathbb{S})$ is equivalent to the unit map of the latter $H\mathbb{Z} \otimes \mathbb{S}[Y] \to H\mathbb{Z} \otimes (\mathbb{S}[X] \square_{\mathbb{S}[X/Y]} \mathbb{S})$, which is the homology equivalence realizing Y as the fiber of the map $X \to X/Y$. Since $M(f \circ i)$ is a connective spectrum, and the limit of a connective cosimplicial spectrum remains connective, we have the equivalence. □

Remark 4.2.3.7. Note that the above proposition would fail in the case that Mf was not $H\mathbb{Z}$-oriented. However, the result would still hold after completing at 2, since the composite morphism $X \to BGL_1(\mathbb{S}) \to BGL_1(H\mathbb{Z}/2)$ is always nullhomotopic for any map $X \to BGL_1(\mathbb{S})$.

Corollary 4.2.3.8. *Let $f : X \to BGL_1(\mathbb{S})$ be a morphism of $\mathbb{E}_n$-monoidal Kan complexes such that $X \to BGL_1(\mathbb{S}) \to BGL_1(H\mathbb{Z})$ is nullhomotopic. Then the induced morphism of $\mathbb{E}_n$-ring spectra $\mathbb{S} \to Mf$ is a Hopf-Galois extension with associated bialgebra $\mathbb{S}[X]$.*

Proof. Apply Theorem 4.2.3.4 to the fibration $* \to X \to X$ and notice that $X \otimes_* *$ remains $\mathbb{E}_n$-monoidal. □

The Lemmas

Lemma 4.2.3.9. *The fiber of the morphism $X \to X/Y$ in the category of $\mathbb{E}_{n-1}$-monoidal Kan complexes is equivalent to X as an $\mathbb{E}_{n-1}$-algebra.*

Proof. From Proposition 3.2.2.1 of [Lur14] we recall that the fiber of a morphism of $\mathbb{E}_{n-1}$-algebra objects is computed in the underlying category of Kan complexes. From Corollary 8.3 of [May75] we recall that the fiber of the map $X \to X/Y$ is indeed equivalent to X (i.e. $x \to X/Y$ is a principal Y-fibration). Thus the fiber of the map of $\mathbb{E}_{n-1}$-monoidal Kan complexes $X \to X/Y$ is equivalent as an $\mathbb{E}_{n-1}$-monoidal Kan complex to Y. □

Lemma 4.2.3.10. *The $\mathbb{E}_{n-1}$-monoidal left Kan extension of $X \to BGL_1(\mathbb{S}) \to \mathcal{S}$ along $X \to X/Y$ is computed by taking the colimit of the composition*

$$fib(X \to X/Y) \to X \to BGL_1(\mathbb{S}) \to \mathcal{S}.$$

Proof. Following the notation given in Definition 3.1.2.2 and the construction in Remark 3.1.3.15 of [Lur14], we have a correspondence of ∞-operads is given by

$$\mathcal{M}^{\otimes} \simeq (X^{\otimes} \times \Delta^1) \coprod_{X^{\otimes} \times \{1\}} X/Y^{\otimes} \to \mathcal{F}in_* \times \Delta^1.$$

In other words, there is a family of ∞-operads indexed by Δ^1 which looks like $X^{\otimes}$ (the ∞-operad associated to X as an $\mathbb{E}_n$-monoidal Kan complex) at one end and $X/Y^{\otimes}$ at the other end. Formula $(*)$ of Definition 3.1.2.2 of [Lur14] states that the value of the desired Kan extension at a 0-simplex $\sigma \in X/Y$ is given by the colimit diagram:

$$((\mathcal{M}^{\otimes}_{act})_{/\sigma} \times_{\mathcal{M}^{\otimes}} X^{\otimes})^{\triangleright} \to (\mathcal{M}^{\otimes})^{\triangleright}_{/\sigma} \to \mathcal{M}^{\otimes} \to \mathcal{T}$$

where the morphism $(\mathcal{M}^{\otimes})^{\triangleright}_{/\sigma} \to \mathcal{M}^{\otimes}$ takes the cone point to σ. In other words, the value of the Kan extension at σ is computed by taking the colimit over the diagram in $\mathcal{M}^{\otimes}$ of objects (and active morphisms) living over σ. As the simplicial set $\mathcal{M}^{\otimes}$ is nothing more than the mapping cylinder of the morphism of $\mathbb{E}_n$-monoidal Kan complexes $X^{\otimes} \to X/Y^{\otimes}$, we have the result. ☐

Lemma 4.2.3.11. *There is a Δ^2-family of ∞-operads induced by the morphisms of $\mathbb{E}_{n-1}$-monoidal Kan complexes $X \to X/Y$ and $X/Y \to *$, denoted $\mathcal{M}^{\otimes} \to \Delta^2 \times \mathcal{F}in_*$, and the induced projection $\mathcal{M}^{\otimes} \to \Delta^2$ is a flat categorical fibration.*

Proof. The composition $(X \to X/Y \to X) \simeq (X \to *)$ is given by a 2-simplex in the quasicategory of $\mathbb{E}_{n-1}$-monoidal quasicategories, hence by a morphism of simplicial sets in $Hom(\Delta^2, Hom(\mathbb{E}^{\otimes}_{n-1}, q\mathcal{C}at)) \simeq Hom(\Delta^2 \times \mathbb{E}^{\otimes}_{n-1}, q\mathcal{C}at)$. By the quasicategorical Grothendieck construction of [Lur09], we obtain a coCartesian fibration of simplicial sets $p : \mathcal{M}^{\otimes} \to \Delta^2 \times \mathbb{E}^{\otimes}_{n-1}$ such that $p^{-1}(0) \simeq X^{\otimes}$, $p^{-1}(1) \simeq X/Y^{\otimes}$ and $p^{-1}(2) \simeq *^{\otimes}$, where $X^{\otimes}$, $X/Y^{\otimes}$ and $*^{\otimes}$ are the ∞-operads witnessing the $\mathbb{E}_{n-1}$-monoidal structure on X, X/Y and $*$. The projection map induces a family of ∞-operads $\mathcal{M}^{\otimes} \to \Delta^2$. This projection is a flat fibration as it satisfies the requirements of Example B.3.4 of [Lur14], i.e. there are coCartesian lifts of every edge in $\Delta^2 \simeq \Delta^2 \times * \subset \Delta^2 \times \mathcal{F}in_*$. ☐

The following proposition is relatively important to the main theorem of this note, so we will explain the intuition behind it first. We have two constructions that we wish to show are equivalent:

1. the Kan extension of $X \to BGL_1(\mathbb{S}) \to \mathcal{S}$ along $X \to X/Y$, which looks like:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & BGL_1(\mathbb{S}) \arrow[r, "c"] & \mathcal{S} \\ X/Y \arrow[urr, "\psi"'] & & \end{tikzcd}$$

2. and the composition $\phi : X/Y \to BGL_1(M(f \circ i)) \to LMod_{M(f \circ i)} \to \mathcal{S}$ where the first map comes from the universal property of X/Y and the trivial H-action on $BGL_1(M(f \circ i))$:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & BGL_1(\mathbb{S}) \arrow[r, "t"] & BGL_1(M(f \circ i)) \arrow[r, "c'"] & LMod_{M(f\circ i)} \arrow[r, "u"] & \mathcal{S} \\ X/Y. \arrow[urr] \arrow[urrrr, "\phi"'] & & & & \end{tikzcd}$$

The former, the Kan extension, is computable by Lemma 4.2.3.10 above, and thus it can be identified as picking out a $\Omega X/Y$-action on $Mf \simeq \mathbb{S}/\Omega Y$ relative to the ΩX-action on $\mathbb{S}$. The latter has the desired property of yielding a colimit whose target is a Thom spectrum over $M(f \circ i)$, and as such supports a Thom isomorphism and Thom diagonal. Hence by showing that the two functors are equivalent we are able to see that Mf is a relative Thom spectrum over $M(f \circ i)$.

Proposition 4.2.3.12. *The Kan extension of $X \to BGL_1(\mathbb{S}) \to \mathcal{S}$ along $X \to X/Y$ is equivalent as an $\mathbb{E}_{n-1}$-monoidal morphism to a morphism that factors as a morphism of $\mathbb{E}_{n-1}$-monoidal Kan complexes through $BGL_1(M(f \circ i))$.*

Proof. We prove the proposition by constructing a composition $X \to X/Y \to BGL_1(M(f \circ i))$ and proving that it is equivalent (after including into $LMod_{M(f\circ i)}$ and forgetting down to $\mathcal{S}$) to the Kan extension. Since $M(f \circ i)$ is a colimit of $Y \to X \to BGL_1(\mathbb{S}) \to \mathcal{S}$, $M(f \circ i)$ must be $M(f \circ i)$-oriented. Thus the composition $Y \to X \to BGL_1(\mathbb{S}) \to BGL_1(M(f \circ i)$ is null homotopic. Hence the induced action of Y on $BGL_1(M(f \circ i))$ is trivial, and by Lemma 4.2.3.13 there is a factorization of $\mathbb{E}_{n-1}$-monoidal Kan complexes $X \to X/Y \to BGL_1(M(f \circ i))$. Let ϕ denote the composition $X/Y \to BGL_1(M(f \circ i)) \hookrightarrow LMod_{M(f\circ i)} \to \mathcal{S}$ and let ψ denote the Kan extension $X/Y \to \mathcal{S}$ described in the proof of Theorem 4.2.3.4. By Corollary 3.1.3.4 of [Lur14] we know that ψ, as a left Kan extension, is produced as a left adjoint. In other words, there is an adjunction

$$Lan_q : Alg_X(\mathcal{S}) \leftrightarrows Alg_{X/Y}(\mathcal{S}) : \Theta$$

where the right adjoint Θ is given by composition with the quotient map $q : X \to X/Y$ and the left adjoint Lan_q is given by taking the left operadic Kan extension along q (here the operad in question is $\mathbb{E}_{n-1}^{\otimes}$, but for simplicity we leave it out of the notation). Hence we have an equivalence of mapping spaces:

$$Alg_X(\mathcal{S})(c \circ f, \phi \circ q) \simeq Alg_{X/Y}(\mathcal{S})(\psi, \phi)$$

where $c : BGL_1(\mathbb{S}) \hookrightarrow \mathcal{S} \simeq LMod_{\mathbb{S}}$ is the canonical inclusion (note that on the left hand side we are implicitly using the fact that $\psi \circ q \simeq c \circ f$ by the definition of a Kan extension). By the natural equivalence of Lemma 4.2.3.13, we obtain an equivalence of mapping spaces

$$Alg_{X/Y}(\mathcal{S})(\psi, \phi) \simeq Alg_X(\mathcal{S})(c \circ f, u \circ c' \circ t \circ f)$$

where $u \circ c' \circ t$ is the composition

$$BGL_1(\mathbb{S}) \xrightarrow{t} BGL_1(M(f \circ i)) \xhookrightarrow{c'} LMod_{M(f \circ i)} \xrightarrow{u} \mathcal{S},$$

t being given by tensoring with $M(f \circ i)$ and u being the forgetful functor. In case it is not clear that $u \circ c' \circ t$ preserves $\mathbb{E}_{n-1}$-algebras, notice that it is the application of the monad associated to $M(f \circ i)$ as an $\mathbb{E}_n$-ring spectrum, so it is at least $\mathbb{E}_{n-1}$ lax monoidal (by Corollary 7.3.2.7 of [Lur14]). Now we have two morphisms of $\mathbb{E}_{n-1}$-monoidal quasicategories: $c \circ f$ and $u \circ c' \circ t \circ f$, but the latter is precisely the former composed with the application of extension/restriction of scalars adjunction associated to $M(i \circ f)$. Hence the application of the unit of this monad, which is a natural transformation $id_{\mathcal{S}} \Rightarrow u \circ t$, induces a natural transformation (of $\mathbb{E}_{n-1}$-monoidal functors) $c \circ f \Rightarrow u \circ t \circ f$. Passing back along the adjunction $Lan_q \dashv \Theta$, we obtain a morphism of $\mathbb{E}_{n-1}$-algebras $\psi \to \phi$. Finally, noticing that this morphism is an equivalence on objects, we have that it is a natural equivalence. Thus ϕ and ψ are equivalent. Noticing that ϕ factors through $BGL_1(M(f \circ i)$ we have proven the proposition. □

Lemma 4.2.3.13. *Given a morphism of $\mathbb{E}_{n-1}$-monoidal Kan complexes $X \to Z$ such that the Y-module structure on Z induced by the composition $Y \xrightarrow{i} X \to Z$ is the trivial Y-module structure, there is a factorization $X \to X/Y \to Z$ and a natural equivalence of functors between $X \to X/Y \to Z$ and $X \to Z$.*

Proof. The quotient X/Y is constructed as the relative tensor product $X \otimes_Y *$ (see Definition 4.2.3.2). By using the $\mathbb{E}_{n-1}$-Y-algebra structure of X and $*$ (since X and Y both receive $\mathbb{E}_n$-

monoidal morphisms from Y), we can construct X/Y as a bar construction in $\mathbb{E}_{n-1}$-Y-algebras (as in Theorem 4.4.2.8 of [Lur14]). Since the Y-action on Z is trivial and is induced by the composition $Y \to X \to Z$, the morphism $X \to Z$ then defines a morphism from the simplicial object defining X/Y into the constant simplicial object on Z. Or, in other words, there is a coherently Y-bilinear morphism $X \otimes_Y * \to Z$ essentially coming from the commutative diagram

$$\begin{tikzcd}
X \times Y \times * \arrow[r] \arrow[d] & X \times * \arrow[d] \\
X \times * \arrow[r] & Z
\end{tikzcd}$$

where the upper horizontal map and the left-hand vertical map are the Y-action on X and the unit of Y, respectively, and the other maps are the given map $X \to Z$. Thus by the universal property of the colimit (again being taken within $\mathbb{E}_{n-1}$-monoidal Kan complexes) the needed factorization is obtained. □

Examples

A large number of morphisms of $\mathbb{E}_n$-monoidal Kan complexes fit into the framework described in the introduction and Theorem 4.2.3.4. We assume that a morphism of simply connected n-fold loop spaces is always modeled by a morphism of $\mathbb{E}_n$-monoidal Kan complexes with a unique 0-simplex (i.e. reduced Kan complexes). The following Lemma allows us to positively identify such examples:

Proposition 4.2.3.14. *Let $F \to E$ be a morphism of connected $\mathbb{E}_n$-monoidal Kan complexes for $n \geq 1$ and let X be the $\mathbb{E}_{n-1}$-monoidal fiber of the induced $\mathbb{E}_{n-1}$-morphism $BF \to BE$. Then $F \to E \to X$ satisfies the conditions of Theorem 4.2.3.4. In particular X is equivalent to E/F as an $\mathbb{E}_{n-1}$-monoidal Kan complex.*

Proof. The universal property of E/F (cf. Lemma 4.2.3.13) induces a morphism of $\mathbb{E}_{n-1}$-monoidal Kan complexes $E/F \to X$, which induces an equivalence on underlying Kan complexes (see e.g. [May75]). □

The following table gives a number of interesting examples of this structure:

Fibration	Hopf-Galois Extension	Bialgebra
$B\mathrm{SU} \to B\mathrm{U} \to \mathbb{C}P^\infty$	$M\mathrm{SU} \to M\mathrm{U}$	$\mathbb{S}[\mathbb{C}P^\infty]$
$B\mathrm{String} \to B\mathrm{Spin} \to K(\mathbb{Z}, 4)$	$M\mathrm{String} \to M\mathrm{Spin}$	$\mathbb{S}[K(\mathbb{Z}, 4)]$
$B\mathrm{U} \to B\mathrm{SO} \to \mathrm{Spin}$	$M\mathrm{U} \to M\mathrm{SO}$	$\mathbb{S}[\mathrm{Spin}]$
$B\mathrm{Sp} \to B\mathrm{SO} \to B(\mathrm{SO}/\mathrm{Sp})$	$M\mathrm{Sp} \to M\mathrm{SO}$	$\mathbb{S}[B(\mathrm{SO}/\mathrm{Sp})]$
$\Omega\mathrm{SU}(n) \to \Omega\mathrm{SU}(n+1) \to \Omega S^{2n+1}$	$X(n) \to X(n+1)$	$\mathbb{S}[\Omega S^{2n+1}]$
$B\mathrm{String} \to B\mathrm{U}[6, \infty) \to B^3\mathrm{Spin}$	$M\mathrm{String} \to M\mathrm{U}[6, \infty)$	$\mathbb{S}[B^3\mathrm{Spin}]$
$B\mathrm{SO} \to BO \to \mathbb{Z}/2$	$M\mathrm{SO} \to MO$	$\mathbb{S}[\mathbb{R}P^\infty]$
$\Omega^2 S^3[3, \infty) \to \Omega^2 S^3 \to S^1$	$H\mathbb{Z}_2^\wedge \to H\mathbb{Z}/2$	$\mathbb{S}[S^1]$

Remark 4.2.3.15. A few entries from the above table may require some explanation:

1. The spectra $X(n)$ were defined by Ravenel in [Rav86] and play an essential role in Devinatz, Hopkins and Smith's proof of Ravenel's Nilpotence Conjecture [DHS88] [HS98].

2. The fibration $B\mathrm{String} \to B\mathrm{U}[6, \infty) \to B^3\mathrm{Spin}$ is perhaps not well known to many readers and can be found in [KLW04] along with many other interesting fibrations.

3. Notice that the final two Hopf-Galois extensions above are only Hopf-Galois extensions over the 2-complete sphere but in light of Remark 4.2.3.7, Theorem 4.2.3.4 can be applied with minor changes.

Remark 4.2.3.16. Perhaps some of the most useful consequences of the above types of identifications are "torsor" type equivalences. In other words given a Hopf-Galois extension $A \to B$ with associated bialgebra H, we have an equivalence $B \otimes_A B \simeq B \otimes H$. The above table thus yields the following equivalences:

1. $M\mathrm{U} \otimes_{M\mathrm{SU}} M\mathrm{U} \simeq M\mathrm{U} \otimes \mathbb{S}[\mathbb{C}P^\infty]$

2. $M\mathrm{Spin} \otimes_{M\mathrm{String}} M\mathrm{Spin} \simeq M\mathrm{Spin} \otimes \mathbb{S}[K(\mathbb{Z}, 4)]$

3. $M\mathrm{SO} \otimes_{M\mathrm{U}} M\mathrm{SO} \simeq M\mathrm{SO} \otimes \mathbb{S}[\mathrm{Spin}]$

4. $M\mathrm{SO} \otimes_{M\mathrm{Sp}} M\mathrm{SO} \simeq M\mathrm{SO} \otimes \mathbb{S}[B(\mathrm{SO}/\mathrm{Sp})]$

5. $X(n+1) \otimes_{X(n)} X(n+1) \simeq X(n+1) \otimes \mathbb{S}[\Omega S^{2n+1}]$

6. $M\mathrm{U}[6,\infty) \otimes_{M\mathrm{String}} M\mathrm{U}[6,\infty) \simeq M\mathrm{U}[6,\infty) \otimes \mathbb{S}[B^3\mathrm{Spin}]$

7. $M\mathrm{SO} \otimes_{MO} M\mathrm{SO} \simeq M\mathrm{SO} \otimes \mathbb{S}_2^\wedge[\mathbb{R}P^\infty]$

8. $H\mathbb{Z}/2 \wedge_{H\mathbb{Z}_2^\wedge} H\mathbb{Z}/2 \simeq H\mathbb{Z}/2 \wedge \mathbb{S}_2^\wedge[S^1]$

Remark 4.2.3.17. Some of the examples in Remark 4.2.3.16 can be verified by traditional computations using the spectral sequence of Theorem 6.4 [EKMM95]:

$$Tor_{p,q}^{E_*(R)}(E_*(M), E_*(N)) \Rightarrow E_{p+q}(M \otimes_R N).$$

For instance, for $E = H\mathbb{Z}$, we can relatively easily check that

$$H_*(X(n+1) \otimes_{X(n)} X(n+1); \mathbb{Z}) \cong H_*(X(n+1); \mathbb{Z}) \otimes_{\mathbb{Z}} H_*(\Omega S^{2n+1}; \mathbb{Z}).$$

Similar computations can be made for $M\mathrm{U}$ over $M\mathrm{SU}$ as well as for the fibrations appearing in Bott periodicity. Much of the relevant algebra for the latter has in fact already been determined in [Car60].

Remark 4.2.3.18. In the case that the fibration of interest is a fibration of $\mathbb{E}_\infty$-monoidal Kan complexes, the theorem can be expressed in a different way. Note that given a fibration $Y \to X \to X/Y$, with associated Hopf-Galois extension $MY \to MX$, Theorem 4.2.3.4 allows us to represent MX as $MY/(X/Y)$. In the case that all the spectra involved are $\mathbb{E}_\infty$-rings, this quotient can be presented as an actual pushout in the quasicategory of $\mathbb{E}_\infty$-rings. Thus, for instance, using the same numbering as above, we obtain the following equivalences:

1. $M\mathrm{U} \simeq M\mathrm{SU} \wedge_{S^1} \mathbb{S}$

2. $M\mathrm{Spin} \simeq M\mathrm{String} \wedge_{K(\mathbb{Z},3)} \mathbb{S}$

3. $M\mathrm{SO} \simeq M\mathrm{U} \wedge_{\mathrm{SO}/\mathrm{U}} \mathbb{S}$

4. $M\mathrm{SO} \simeq M\mathrm{Sp} \wedge_{\mathrm{SO}/\mathrm{Sp}} \mathbb{S}$

6. $M\mathrm{U}[6,\infty) \simeq M\mathrm{String} \wedge_{BB\mathrm{Spin}} \mathbb{S}$

These results may be interesting to homotopy theorists interested in doing computations, as in some cases they related hard to understand spectra, e.g. MString, to much more easily understood spectra, e.g. $MU[6, \infty)$.

A New Construction of MU

The contents of this section were in fact the original motivation for the work in this thesis. The author, interested in understanding the underlying derived algebraic geometry of the proofs of Ravenel's conjectures (cf. [DHS88] [HS98]), noticed that the sequence

$$\mathbb{S} \to \ldots \to X(n-1) \to X(n) \to \ldots \to MU$$

looked a great deal like an infinite sequence of Hopf-Galois extensions obtained by some piece of a Galois correspondence from sub-Hopf-algebras of $\mathbb{S}[BU]$ to intermediate extensions of $\mathbb{S} \to MU$. It is of course a consequence of 4.2.3.4 above that this is precisely what is happening. If we allow ourselves a moment of fantasy, and believe that there is some kind of functor $Spec(-)$ that takes $\mathbb{E}_2$-ring spectra to some version of "spectral schemes," then saying that $X(n) \to X(n+1)$ is a $\mathbb{S}[\Omega S^{2n+1}]$-Hopf-Galois extension means that $Spec(X(n+1)) \to Spec(X(n))$ is a principal $Spec(\mathbb{S}[\Omega S^{2n+1}])$-bundle. In other words, $Spec(X(n+1))$ is a twisted tensor product of $Spec(X(n))$ and $Spec(\mathbb{S}[\Omega S^{2n+1}])$. It is work in progress to use this interpretation to better understand the Nilpotence and Periodicity Theorems of Devinatz, Hopkins and Smith, or even provide an alternative proof.

The construction of MU given below is entirely canonical, as it is determined by the preexisting fibrations $\Omega SU(n) \to \Omega SU(n+1) \to \Omega S^{2n+1}$. This determinacy can be made even more explicit by using work of Antolin-Camarena and Barthel [ACB14]. Each $X(n+1)$ is in fact produced from $X(n)$ by attaching an $\mathbb{E}_1$-cell along a canonical element in $\pi_{2n-1}(X(n))$. This construction should be compared to Lazard's construction of the Lazard ring in [Laz75]. Antolin-Camarena and Barthel also have ongoing work applying these concepts to the p-complete Hopf-Galois extension $H\mathbb{Z}_p^\wedge \to H\mathbb{Z}/p$.

Corollary 4.2.3.19. *Let $X(n)$ be the Thom spectrum associated to the morphism of $\mathbb{E}_2$-monoidal Kan complexes $\Omega \mathrm{SU}(n) \to B\mathrm{U} \to BGL_1(\mathbb{S})$. Then $X(n+1)$ is a versal $\mathbb{E}_1$-algebra over $X(n)$ of characteristic $\widehat{\chi}_n$ where $\widehat{\chi}_n$ is a canonical class in $\pi_{2n-1}(X(n))$.*

Proof. Given the fibration $\Omega\mathrm{SU}(n) \to \Omega\mathrm{SU}(n+1) \to \Omega S^{2n+1}$, and an application of Theorem 4.2.3.4 and Lemma 4.2.3.14 above, we can identify $X(n+1)$ as the $\mathbb{E}_1$-monoidal Thom spectrum given by the $\mathbb{E}_1$-monoidal left Kan extension $\Omega S^{2n+1} \to BGL_1(X(n))$. The map of $\mathbb{E}_1$-monoidal Kan complexes

$\chi_n \in Map_{\mathbb{E}_1}(\Omega S^{2n+1}, BGL_1(X(n)))$, by application of standard adjunctions, induces a map of Kan complexes $\tilde{\chi}_n \in Map_{\mathcal{T}}(S^{2n-1}, GL_1(X(n)))$. Note that $\tilde{\chi}_n$ must have image contained in a connected component $u \in \pi_0(GL_1(X(n)) \simeq \mathbb{Z}/2$ which induces a translation $\tau_u : \Omega^\infty X(n) \to \Omega^\infty X(n)$. The composition $\tau_u \circ \tilde{\chi}_n : S^{2n-1} \to \Omega^\infty X(n)$ lifts to a morphism of spectra $\widehat{\chi}_n : \mathbb{S}^{2n-1} \to X(n)$. An application of Theorem 4.10 of [ACB14] gives that $X(n+1)$ is the versal $\mathbb{E}_1$-algebra of characteristic $\widehat{\chi}_n$ on $X(n)$. □

Remark 4.2.3.20. The content of [ACB14] allows us to consider $X(n+1)$ as the $\mathbb{E}_1$-spectrum obtained by attaching an $\mathbb{E}_1$-cell to $X(n)$ along the map $\widehat{\chi}_n$ described above. Note that $\widehat{\chi}_1 \simeq \eta$, the Hopf element in $\pi_1(\mathbb{S})$.

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Curriculum Vitae

Jonathan Beardsley was born in Annapolis, Maryland on April 25th, 1987. He received his bachelor's degree in mathematics from the University of Central Florida in 2010. He graduated magna cum laude with honors in the major, and a minor in computer science. His undergraduate thesis showed that a certain class of generalized functions satisfied descent on Euclidean space of arbitrary dimension. He was also accepted into the doctoral program in the Mathematics Department at Johns Hopkins University in 2010. He received his Masters Degree in mathematics from Johns Hopkins in 2011. His dissertation was completed under the guidance of Jack Morava and defended on February 29th, 2016.